A Crash Course in Integration in $\mathbb{R}^2$ and $\mathbb{R}^3$

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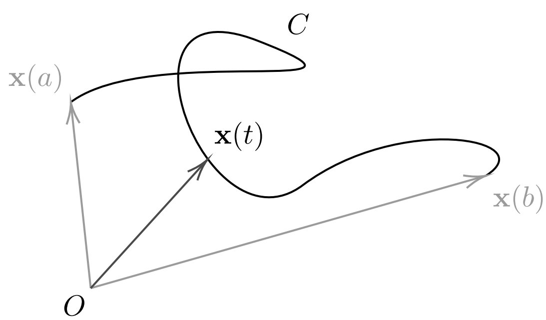
This is a no-frills description of how to integrate over curves, areas, volumes and surfaces in $\mathbb{R}^2$ and $\mathbb{R}^3$. Very little will be actually motivated, but it should (hopefully) still be helpful.

1 The Line Integral

Suppose you have a regular curve C, parametrised by some function $\mathbf{x}(t)$, where $a \leq t \leq b$, and you want to integrate some other function $f : \mathbb{R}^3 \to \mathbb{R}$ over the curve.



Then we have, integrating from $\mathbf{x}(a)$ to $\mathbf{x}(b)$,

$$\int_C f(\mathbf{x}) \mathrm{d}s = \int_a^b f(\mathbf{x}(t)) \left| \frac{\mathrm{d}\mathbf{x}(t)}{\mathrm{d}t} \right| \mathrm{d}t.$$

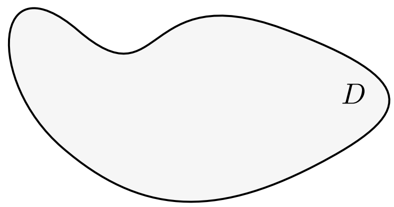
If instead we wanted to integrate over some vector field (by say taking the dot product of that vector field with the unit tangent), then since $\mathrm{d}\mathbf{x}(t)/\mathrm{d}t$ is tangent to the curve, we would have

$$\int_C F(\mathbf{x}) \cdot \mathrm{d}\mathbf{x} = \int_a^b \mathbf{F}(\mathbf{x}(t)) \cdot \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} \mathrm{d}t,$$

where again we integrate from $\mathbf{x}(a)$ to $\mathbf{x}(b)$.

2 Area and Volume Integrals

Suppose you have some closed region $D \subset \mathbb{R}^2$, and you wanted to integrate some function $f : \mathbb{R}^2 \to \mathbb{R}$ over the region.



Then integrate over say the x coordinate first (where the bounds of the inner integral will depend on y), we have

$$\int\int_D f(\mathbf{x})\mathrm{d}A = \int\int_{(x,y)\in D} f(x,y)\mathrm{d}x\ \mathrm{d}y$$

Of course we could have integrated over the y coordinate first, and would probably give the same result.

We can also do a change of coordinate. Suppose we have $x = x(u,v)$ and $y = y(u,v)$ being smooth bijections (with smooth inverse) mapping D' in the (u,v)-plane to the region D in the (x,y)-plane. Then we have

$$\int\int_{(x,y)\in D} f(x,y)\mathrm{d}x\ \mathrm{d}y = \int\int_{(u,v)\in D'} f(x(u,v),y(u,v))|J|\mathrm{d}u\ \mathrm{d}v,$$

where

$$|J| = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \left|\det\left(\frac{\partial \mathbf{x}}{\partial u}\ \middle|\ \frac{\partial \mathbf{x}}{\partial v}\right)\right| = \left|\det\begin{pmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{pmatrix}\right|$$

is the *Jacobian*.

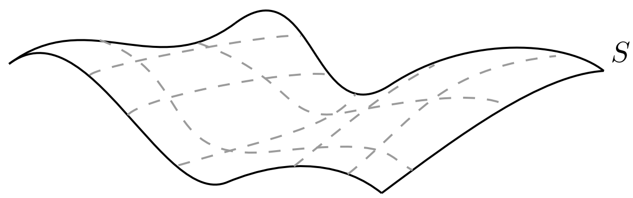
Basically the exact same thing is true for volume integrals – I won't bore you with the details. Just use three variables and change the Jacobian accordingly.

3 Surface Integrals

Suppose you have a surface S, parametrised with

$$S = \{\mathbf{x}(u, v) \mid (u, v) \in D\},$$

for some region D in the $(u,\ v)$-plane. Suppose also that you wanted to integrate some function $f : \mathbb{R}^3 \to \mathbb{R}$ over the surface.



Then we would have

$$\int\int_S f(\mathbf{x})\mathrm{d}S = \int\int_D f(\mathbf{x}(u, v)) \left| \frac{\partial \mathbf{x}}{\mathrm{d}u} \times \frac{\partial \mathbf{x}}{\partial v} \right| \mathrm{d}u\ \mathrm{d}v.$$

If instead we wanted to integrate over some vector field $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ by taking the dot product of the vector field with the unit normal (a flux integral), then since $\frac{\partial \mathbf{x}}{\mathrm{d}u} \times \frac{\partial \mathbf{x}}{\partial v}$ is normal to the curve, we would have

$$\int\int_S \mathbf{F}(\mathbf{x}) \cdot \mathrm{d}\mathbf{S} = \int\int_D \mathbf{F}(\mathbf{x}(u, v)) \cdot \left(\frac{\partial \mathbf{x}}{\mathrm{d}u} \times \frac{\partial \mathbf{x}}{\partial v} \right) \mathrm{d}u\ \mathrm{d}v.$$

It's worth noting here that by doing this, we implicitly chose a direction for the normal (there's 2!). If we wanted the other direction, we can just negate the integral.

4 Integral Theorems

Having read that, you are clearly now an expert integrator, keen to speed up your multidimensional calculations. Luckily we have a few results at our disposal which we will state (naturally) without proof. Throughout, we will use the notation ∂A to denote the boundary of A.

Theorem 4.1 (Conservative Vector Fields)**.** *Suppose the vector field* $\mathbf{F}$ *can be written as* $\mathbf{F} = \nabla f$ *for some scalar function* f*. Then if* C *is any curve from* $\mathbf{x}(a)$ *to* $\mathbf{x}(b)$*, we have*

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{x} = f(\mathbf{x}(b)) - f(\mathbf{x}(a)).$$

Theorem 4.2 (Green's Theorem)**.** $P = P(x, y)$ *and* $Q = Q(x, y)$ *are continuously differentiable functions on* A *and* ∂A *is made from a collection of piecewise smooth curves, then*

$$\int_{\partial A} P \ \mathrm{d}x + Q \ \mathrm{d}y = \iint_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathrm{d}x \ \mathrm{d}y.$$

The orientation of the boundary ∂A *is such that* A *lies to your left as you traverse it.*

Theorem 4.3 (Stokes' Theorem)**.** *If* $\mathbf{F} = \mathbf{F}(\mathbf{x})$ *is a continuously differentiable vector field and* S *is an orientable, piecewise regular surface with piecewise regular boundary* ∂S *then*

$$\int_S (\nabla \times \mathbf{F}) \cdot \mathrm{d}\mathbf{S} = \int_{\partial S} \mathbf{F} \cdot \mathrm{d}\mathbf{x}.$$

Theorem 4.4 (Divergence Theorem in $\mathbb{R}^3$)**.** *If* $\mathbf{F} = \mathbf{F}(\mathbf{x})$ *be a continuously differentiable vector field and* V *is a volume with a piecewise regular boundary* ∂V *then*

$$\int_V \nabla \cdot \mathbf{F} \ \mathrm{d}V = \int_{\partial V} \mathbf{F} \cdot \mathrm{d}\mathbf{S}$$

where the normal to ∂V *points outwards from* V*.*

Theorem 4.5 (Divergence Theorem in $\mathbb{R}^2$)**.** *Let* $\mathbf{F} = \mathbf{F}(\mathbf{x})$ *be a continuously differentiable vector field and* $D \subset \mathbb{R}^2$ *region with piecewise smooth boundary* ∂D *then*

$$\int_D \nabla \cdot \mathbf{F} \ \mathrm{d}A = \oint_{\partial D} \mathbf{F} \cdot \mathbf{n} \mathrm{d}s$$

where the unit normal $\mathbf{n}$ *to* ∂D *points outwards from* D*.*