Methods

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This article constitutes my notes for the 'Methods' course, held in Michaelmas 2021 at Cambridge. These notes are *not a transcription of the lectures*, and differ significantly in quite a few areas. Still, all lectured material should be covered.

Contents

§1 Fourier Series

§1.1 Periodic Functions

We will begin our study of method and in particular Fourier series by considering some periodic functions.

Definition 1.1 (Perioidic)
A function $f(x)$ is **periodic** if $f(x+T) = f(x)$ for all x, where T is the **period**.

Example 1.2 (Simple Harmonic Motion)
Many physical objects are described by *simple harmonic motion*, with the position given by

$$y = A \sin \omega t.$$

We call A the **amplitude**, and the period is $T = 2\pi/\omega$. The **frequency** is $1/T$.

Fourier series is all about trying to write periodic functions as particular sums of sines and cosines. Consider the set of functions

$$g_n(x) = \cos \frac{n\pi x}{L}, \quad \text{and} \quad h_n(x) = \sin \frac{n\pi x}{L},$$

where we take $n \in \mathbb{R}^+$. These functions are periodic on the interval $[0, 2L]$.

You may recall the following set of identities:

$$\begin{aligned}
\cos A \cos B &= \frac{1}{2}\left(\cos(A-B) + \cos(A+B)\right) \\
\sin A \sin B &= \frac{1}{2}\left(\cos(A-B) - \cos(A+B)\right) \\
\sin A \cos B &= \frac{1}{2}\left(\sin(A-B) + \sin(A+B)\right).
\end{aligned}$$

We are going to try and define an inner product on this domain $[0, 2L]$, and using that we will by able to multiply these functions together and talk about their relative orthogonality.

Definition 1.3

We define the inner product $\langle f, g\rangle = \int_0^{2L} f(x)g(x)\,\mathrm{d}x$.

We can then obtain some orthogonality conditions for h_n and g_n with respect to this inner product. We can compute for $n \neq m$

$$\begin{aligned}
\langle h_n, h_m\rangle &= \int_0^{2L} \sin\frac{n\pi x}{L} \sin\frac{m\pi x}{L}\,\mathrm{d}x \\
&= \frac{1}{2}\int_0^{2L} \left(\cos\frac{(n-m)\pi}{L}x - \cos\frac{(n+m)\pi}{L}x\right)\mathrm{d}x \\
&= \frac{1}{2}\frac{L}{\pi}\left[\frac{\sin(n-m)\pi x/L}{n-m} - \frac{\sin(n+m)\pi x/L}{n-m}\right]_0^{2L} \\
&= 0,
\end{aligned}$$

and for $n = m$

$$\begin{aligned}
\langle h_n, h_n\rangle &= \int_0^{2L} \sin^2\frac{n\pi x}{L}\,\mathrm{d}x \\
&= \int_0^{2L} \frac{1}{2}\left(1 - \cos\frac{2\pi n x}{L}\right)\mathrm{d}x \\
&= L.
\end{aligned}$$

Hence we obtain the orthogonality condition

$$\langle h_n, h_m\rangle = \begin{cases} L\delta_{mn} & \text{if } n, m \neq 0, \\ 0 & \text{if } m = 0. \end{cases}$$

Similarly, it's straightforward to check that

$$\langle g_n, g_m\rangle = \begin{cases} L\delta_{mn} & \text{if } n, m \neq 0, \\ 2L\delta_{0n} & \text{if } m = 0. \end{cases}$$

and

$$\langle h_n, g_m\rangle = 0.$$

These orthogonality conditions are important because we are going to use these functions as a complete orthogonal set which spans the space of 'well-behaved periodic functions'.

§1.2 Definition of a Fourier Series

We can express any 'well-behaved' periodic function $f(x)$ with period $2L$ as

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L},$$

where a_n, b_n are constants such that the RHS is convergent for all x where f is continuous. At a discontinuity, the Fourier series approaches the midpoint of the upper and lower limits at that point.

Consider taking the inner product $\langle h_n, f\rangle$ and substitute the expression for f above, to get

$$\int_0^{2L} \sin\frac{m\pi x}{L} f(x)\,\mathrm{d}x = \sum_{n=1}^{\infty} L b_n \delta_{nm} = L b_m.$$

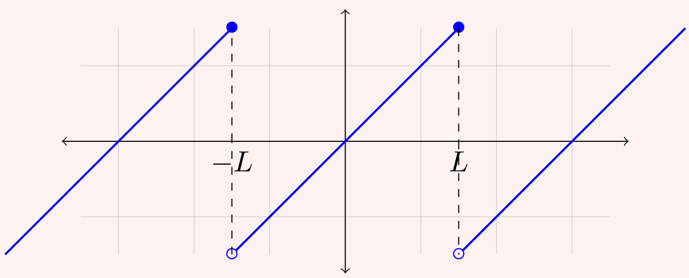
Hence we find that (doing something similar with g_n)

$$\begin{aligned} b_n &= \frac{1}{L}\int_0^{2L} f(x)\sin\frac{n\pi x}{L}\,\mathrm{d}x, \\ \text{and}\quad a_n &= \frac{1}{L}\int_0^{2L} f(x)\cos\frac{n\pi x}{L}\,\mathrm{d}x. \end{aligned}$$

Now, this expression for a_n includes the case $n = 0$, and says that it is the average value of the function. Also, the range of integration is one period, and we can equivalently integrate over $[-L, L]$ instead of $[0, 2L]$.

Example 1.4 (The Sawtooth Wave)

Consider the function $f(x) = x$ for $-L \le x \le L$, with the function being periodic elsewhere.



Here we have

$$a_n = \frac{1}{L}\int_{-L}^{L} x\cos\frac{n\pi x}{2}\,\mathrm{d}x = 0, \qquad \text{(integrating an odd function)}$$

for all n, and

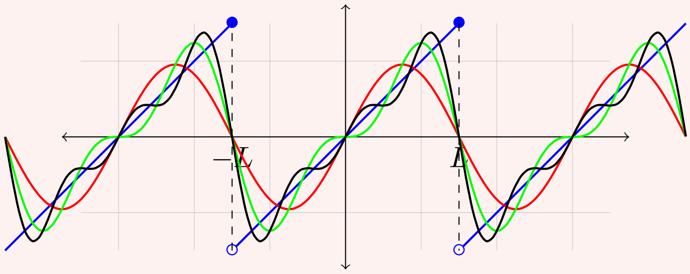
$$\begin{aligned} b_n &= \frac{2}{L}\int_0^L x\sin\frac{n\pi x}{L}\,\mathrm{d}x \\ &= \frac{-2}{n\pi}\left[x\cos\frac{n\pi x}{L}\right]_0^L + \frac{2}{n\pi}\int_0^h \cos\frac{n\pi x}{L}\,\mathrm{d}x \\ &= -\frac{2L}{n\pi}\cos n\pi + \frac{2L}{(n\pi)^2}\cancel{\sin n\pi} \end{aligned}$$

$$= \frac{2L}{n\pi}(-1)^{n+1}.$$

So the sawtooth Fourier series is

$$2L \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n\pi} \sin\left(\frac{n\pi x}{L}\right) = \frac{2L}{\pi}\left[\sin\left(\frac{\pi x}{L}\right) - \frac{1}{2}\sin\left(\frac{2\pi x}{L}\right) + \frac{1}{3}\sin\left(\frac{3\pi x}{L}\right) + \cdots\right].$$

which is slowly convergent.



§1.3 The Dirichlet Conditions (Fourier's Theorem)

So we need 'well behaved' functions for what we have discussed about Fourier series to work, and for a Fourier series for f to be unique. But what exactly does it mean for a function to be 'well behaved' in this context? This is specified by the Dirichlet conditions, also known as Fourier's theorem.

Theorem 1.5 (The Dirichlet Conditions)
If f is a bounded periodic function with period $2L$ and finitely many minima, maxima and discontinuities on $0 \leq x \leq 2L$, then the Fourier series converges to $f(x)$ at all points where f is continuous, and at discontinuities converges to $\frac{1}{2}(f(x_+) + f(x_-))$.

One might note that these conditions are *much weaker* than those needed for Taylor series, but it does eliminate pathological functions such as $\sin 1/x$, $1/x$ and the indicator function on the rationals. The converse of this is *not* true, for example $\sin 1/x$ has a well defined Fourier series.

The proof of this result is *too* difficult, but is best done using complex methods so we will not dwell on it in this course.

The rate of convergence of the Fourier series depends on the 'smoothness' of the function.

Theorem 1.6 (Rate of Convergence of Fourier Series)
If $f(x)$ has continuous derivatives up to a pth derivative which is discontinuous, then the Fourier series converges as $\mathcal{O}(n^{-(p+1)})$ as $n \to \infty$.

Example 1.7 (Square Wave)
Consider the function

$$f(x) \begin{cases} 1 & \text{if } 0 \leq x < 1, \\ -1 & \text{if } -1 \leq x < 0. \end{cases}$$

This function has discontinuous $p = 0$th derivative (the function is discontinuous!), and the fourier series of this function is given by

$$f(x) = 4 \sum_{m=1}^{\infty} \frac{\sin(2m-1)\pi x}{(2m-1)\pi}.$$

Example 1.8 ('See-Saw' Wave)

Consider the function

$$f(x) \begin{cases} x(1-\xi) & \text{if } 0 \le x < \xi, \\ \xi(1-x) & \text{if } \xi \le x < 1, \\ f(-x) & \text{if } -1 \le x < 0. \end{cases}$$

This function has discontinuous $p = 1$st derivative, and it's Fourier series is given by

$$f(x) = 2 \sum_{m=1}^{\infty} \frac{\sin(n\pi\xi)\sin(n\pi x)}{(n\pi)^2}.$$

Integration of Fourier Series

As a general principle, it is *always* valid to integrate the Fourier series of $f(x)$ term-wise to obtain

$$F(x) = \int_{-x}^{x} f(x)\,\mathrm{d}x,$$

because $F(x)$ satisfies the Dirichlet conditions if $f(x)$ does.

Differentiation of Fourier Series

As a general principle, you must *take care* when differentiating a Fourier series term-wise. Let's look at an example.

Example 1.9 (Differentiating the Square Wave)

Consider the clearly non-differentiable square wave function, defined in Example 1.7. We had

$$f(x) = 4 \sum_{m=1}^{\infty} \frac{\sin(2m-1)\pi x}{(2m-1)\pi}.$$

We can try and differentiate this term-wise to get

$$f'(x) \stackrel{?}{=} 4 \sum_{m=1}^{\infty} \cos(2m-1)\pi x,$$

but this series is unbounded!

So we need to be careful. Here's the result about what we can do with differentiation.

Theorem 1.10
If $f(x)$ is continuous and satisfies the Dirichlet conditions, and $f'(x)$ satisfies the Dirichlet conditions, then $f'(x)$ can be found by term-wise differentiation of the Fourier series of $f(x)$.

§1.4 Parseval's Theorem

Parseval's theorem is a relation between the integral of the square of a function and the square of the Fourier coefficients:

$$\begin{aligned}\int_0^{2L} [f(x)]^2 \,\mathrm{d}x &= \int_0^{2L} \mathrm{d}x \left[\frac{1}{2}a_0 + \sum_n a_k \cos\frac{n\pi x}{L} + \sum_n b_n \sin\frac{n\pi x}{L}\right]^2 \\ &= \int_0^{2L} \mathrm{d}x \left[\frac{1}{4}a_0^2 + \sum_n a_n^2 \cos^2\frac{n\pi x}{L} + \sum_n b_n^2 \sin^2\frac{n\pi x}{L}\right] \\ &= 2\left[\frac{1}{2}a_0^2 + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right)\right].\end{aligned}$$

This result is also called the completeness relation, since the $LHS \geq RHS$ if any of the basis functions are missing.

Example 1.11 (The Basel Problem)
Consider the sawtooth wave $f(x) = x$ on $-L \leq x \leq L$. The the LHS is

$$\int_{-L}^{L} x^2 \,\mathrm{d}x = \frac{2}{3}L^3,$$

and the RHS is

$$L\sum_{n=1}^{\infty}\frac{4L^2}{n^2\pi^2} = \frac{4L^3}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2},$$

and thus $\sum_{n=1}^{\infty} 1/n^2 = \pi^2/6$.

§1.5 Alternative Fourier Series

There are some alternative forms of fourier series.

Half-Range Series

Consider $f(x)$ defined only on $0 \leq x < L$. We can extend it's range over $-L \leq x < L$ in two simple ways:

1. Require f to be odd, with $f(-x) = -f(x)$. Then considering it's Fourier series we would have $a_n = 0$ for all n (as cos is even), and

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{2}\,\mathrm{d}x.$$

This implies that $f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$, which is a **Fourier sine series**.

2. Require f to be even, with $f(-x) = f(x)$, Then $b_n = 0$, and

$$a_n = \frac{2}{L} \int_0^2 f(x) \cos \frac{n\pi x}{L} \, \mathrm{d}x,$$

so $f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L}$, which is a **Fourier cosine series**.

Complex Representation

Recall that we can write

$$\cos \frac{n\pi x}{L} = \frac{1}{2}\left(e^{in\pi x/L} + e^{-in\pi x/L}\right) \quad \text{and} \quad \sin \frac{n\pi x}{L} = \frac{1}{2i}\left(e^{in\pi x/L} - e^{-in\pi x/L}\right).$$

So the Fourier series is

$$\begin{aligned} f(x) &= \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{2} \\ &= \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n - ib_n)\, e^{in\pi x/L} + \sum_{n=1}^{\infty} (a_n + ib_n)\, e^{-in\pi x/L} \\ &= \sum_{m=-\infty}^{\infty} c_m e^{im\pi x/L}, \end{aligned}$$

where for $m > 0$, $c_m = \frac{1}{2}(a_m - ib_m)$ and for $m < 0$, $c_m = \frac{1}{2}(a_{-m} + ib_{-m})$. Equivalently,

$$c_m = \frac{1}{2L} \int_{-L}^{L} f(x) e^{-im\pi x/L} \, \mathrm{d}x.$$

§1.6 Some Fourier Series Motivations – Self-Adjoint Matrices

We will now review a few results about matrices. Suppose that $\mathbf{u}, \mathbf{v} \in \mathbb{C}^n$ with inner product $\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u}^\dagger \mathbf{v}$. The $n \times n$ matrix A is **self-adjoint** or **hermitian** if

$$\langle A\mathbf{u}, v \rangle = \langle u, A\mathbf{v} \rangle,$$

that is, if $A^\dagger = A$.

The eigenvalues λ_i and eigenvectors $\mathbf{v}_i$ satisfy

$$A\mathbf{v}_i = \lambda_i \mathbf{v}_i,$$

and have the following properties

(i) The eigenvalues are real

(ii) If $\lambda_i \neq \lambda_j$, then the corresponding eigenvectors are orthogonal with $\langle \mathbf{v}_i, \mathbf{v}_j \rangle = 0$

(iii) We can rescale to create an orthonormal basis $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$.

Given $\mathbf{b}$, we can solve for $\mathbf{x}$ in the equation

$$A\mathbf{x} = \mathbf{b}.$$

Express $\mathbf{b} = \sum_{i=1}^{n} b_i \mathbf{v}_i$, and we seek a solution $\mathbf{x} = \sum_{i=1}^{n} c_n \mathbf{v}_n$. Substituting this into the above equation, we get

$$A\mathbf{x} = \sum_i A c_i \mathbf{v}_i = \sum_i c_i \lambda_i \mathbf{v}_i = \sum_i b_i \mathbf{v}_i.$$

By orthogonality, we can equate coefficients to get $c_n \lambda_n = b_n$, so our solution is

$$\mathbf{x} = \sum_{i=1}^{n} \frac{b_i}{\lambda_i} \mathbf{v}_i.$$

So once we have a self-adjoint matrix A and have found all of the eigenvalues, we then have a general methodology for solving any matrix equation of the form $A\mathbf{x} = \mathbf{b}$. It turns out that you can do the same with Fourier series!

We can find a general solution to the differential equation for which cosines and sines are the eigenfunctions.

§1.7 Solving Inhomogeneous ODEs with Fourier Series

We wish to find $y(x)$ given $f(x)$ for the differential operator

$$\mathcal{L}[y] \equiv -\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = f(x).$$

To solve this, we need some boundary conditions, which we will take as $y(0) = y(L) = 0$.

The related eigenvalue problem is

$$\mathcal{L}[y_n] = \lambda_n y_n,$$

with $y_n(0) = y_n(L) = 0$, which has eigenfunctions and eigenvalues

$$y_n(x) = \sin \frac{n\pi x}{L}, \quad \lambda_n = \left(\frac{n\pi}{L}\right)^2.$$

We seek solutions as half-range sine series. To do that, we will try substitute $y(x) = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi x}{L}$, and expand $f(x)$ in terms of it's fourier series, $f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$, with $b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} \, \mathrm{d}x$.

Substituting this into the differential equation, we get

$$\mathcal{L}[y] = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} \left(\sum_{n=1}^{\infty} c_n \sin \frac{n\pi x}{L} \right) = \sum_{n=1}^{\infty} c_n \left(\frac{n\pi}{L}\right)^2 \sin \frac{n\pi x}{L}.$$

By orthogonality, we have $c_n \left(\frac{n\pi}{L}\right)^2 = b_n$, so the solution is

$$y(x) = \sum_{n=1}^{\infty} \frac{b_n}{\left(\frac{n\pi}{L}\right)^2} \sin \frac{n\pi x}{L} = \sum_{n=1}^{\infty} \frac{b_n}{\lambda_n} y_n.$$

Example 1.12

Consider the 'square wave' source given by $f(x) = 1$ for $0 \leq x < 1$, with the function being odd. This function has fourier series

$$f(x) = 4\sum_{m=1}^{\infty} \frac{\sin(2m-1)\pi x}{(2m-1)\pi}.$$

So the solution for y must then by

$$y(x) = \sum_n \frac{b_n}{\lambda_n} y_n = 4\sum_m \frac{\sin(2m-1)\pi x}{((2m-1)\pi)^3},$$

but this is the Fourier series for $y(x) = \frac{1}{2}x(1-x)$!