Olympiad Graph Theory

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§1 Foundations

Graph theory is an area of combinatorics which has lots of fun problems and plenty of interesting theorems. It's a common topic in Olympiads and is a very active field of research. In this handout we are going to look at the basics of Graph Theory, with a particular focus on how it's used in solving Olympiad problems.

§1.1 Definitions

We will begin this handout on graph theory naturally by defining what a graph is.

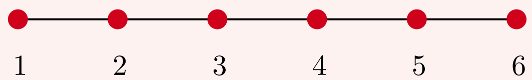
Definition 1.1 (Graph)

A **graph** is an ordered pair $G = (V, E)$ where V is the set of **vertices**, and $E \subseteq \{\{x, y\} \mid x, y \in V, x \neq y\}$ is a set of unordered pairs of vertices called **edges**.

We have a natural way of drawing a graph. For each vertex we have a point in the plane, and for each edge we draw a line between the corresponding pair of vertices.

Example 1.2 (Example of a Graph)

The ordered pair (V, E) where $V = \{1, 2, \ldots, 6\}$ and $E = \{\{1, 2\}, \{2, 3\}, \ldots, \{5, 6\}\}$ is a graph.



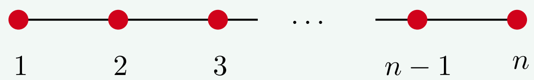
This graph is known as P_6, a path on 6 vertices.

§1.1.1 Common Graphs

There are some graphs that will appear repeatedly when working on problems involving graph theory, and we will define them now.

Definition 1.3 (Path)

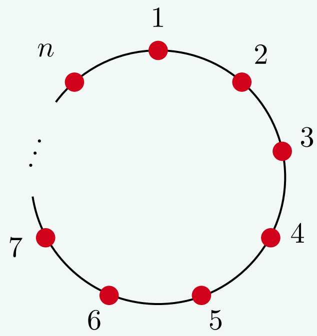
We define P_n to be the graph $V = \{1, \ldots, n\}$, $E = \{\{1, 2\}, \{2, 3\}, \ldots, \{n-1, n\}\}$ as shown.



We call this a **path** on n vertices, and say it has **length** $n-1$.

Definition 1.4 (Cycle)

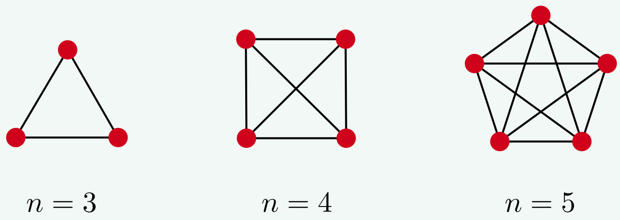
We define C_n (for $n \geq 3$) to be the graph $V = \{1, \ldots, n\}$, $E = \{\{1,2\}, \ldots, \{n-1, n\}, \{n, 1\}\}$ as shown.



We call this the **cycle** on n vertices.

Definition 1.5 (Complete Graph)

The **complete graph** on n vertices K_n is the graph $\{1, \ldots, n\}$ and $E = \{\{i,j\} \mid i \neq j \in V\}$.

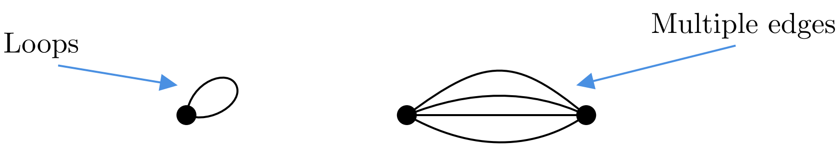


Note that there is an edge between every pair of vertices.

Definition 1.6 (Empty Graph)

We define the **empty graph** on n vertices $\overline{K_n}$ to have $V = \{1, \ldots, n\}$ but $E = \emptyset$.

Remark. In our definition of a graph, we *don't allow*[1] loops, and there *cannot* be multiple edges between the same set of vertices.



You can define graphs where such things are allowed, but for now we will outlaw them. We also note that edges are *unordered pairs*, so for now edges have no direction.

To be slightly more succinct, we will use some shorthand notation.

Notation. If $G = (V, E)$ is a graph, and we have some edge $\{x, y\} \in E$, we will denote

[1] These limitations are inherent in our definition, where we use sets rather than multisets.

it by xy. We will also define $|G| = |V|$, and $e(G) = |E|$.

Example 1.7 (Vertices and Edges of K_n)

Consider the graph K_n. We have $|K_n| = n$, and $e(K_n) = \binom{k}{2}$, as there is an edge between any pair of vertices.

§1.1.2 Subgraphs

Now we will define the notion of a *subgraph*, in the natural way.

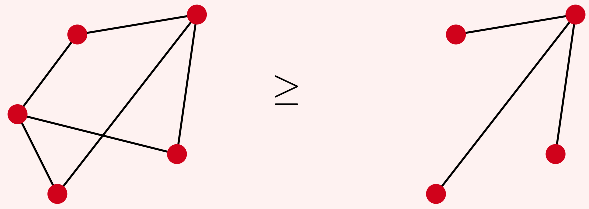
Definition 1.8 (Subgraph)

We say that $H = (V', E')$ is a **subgraph** of $G = (V, E)$ if $V' \subseteq V$ and $E' \subseteq E$.

Informally, H is a subgraph of G if we can remove vertices and edges from G to get H. Let's look at some examples.

Example 1.9 (Example of a Subgraph)

The graph on the right is a subgraph of the graph on the left.



An easy way to get a subgraph is by taking a subset of the vertices and seeing what edges you get from the original graph.

Definition 1.10 (Inducted Subgraph)

If $G = (V, E)$ is a graph and $X \subseteq V$, the **subgraph inducted by** X is defined to be $G[X] = (X, \{xy \in E : x, y \in X\})$.

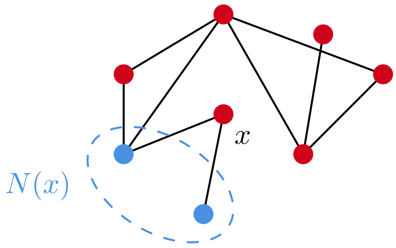
§1.1.3 Neighbors and Degree

Now for the following discussion, fix some graph $G = (V, E)$, and let $x \in V$.

Definition 1.11 (Neighborhood)

If $xy \in E$, then we say that x and y are **adjacent**. We define the **neighborhood** of x to be the set $N(x) = \{y \in V \mid xy \in E\}$ of all vertices adjacent to x.

Note that as in the diagram below, x is not in its own neighborhood.



Definition 1.12 (Degree)

We define the **degree** of a vertex x to be $d(x) = |N(x)|$. This is equal to the number of edges that are incident to x.

Definition 1.13 (Regularity)

A graph G is said to be **regular** if all of the degrees are the same. We say G is k-regular if $d(x) = k$ for all $x \in V$.

Example 1.14 (Regular and Non-Regular Graphs)

The graphs K_n is $n-1$ regular, and C_n is 2-regular. The graph P_n is not regular.

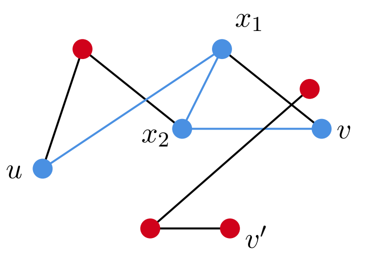
Definition 1.15 (Minimum/Maximum Degree)

Let G be a graph. The **maximum degree** of G, $\triangle(G)$ is defined to be $\triangle(G) = \max_{x \in V} d(x)$. Similarly, we define the **minimum degree** of G, $\delta(G)$ to be $\delta(G) = \min_{x \in V} d(x)$.

In a k-regular graph as mentioned above, we have $\triangle(G) = \delta(G) = k$.

§1.1.4 Connectivity

We now want to define some notion of *connectivity*.



For example, in the graph above we want to say somehow that u and v are connected, but u and v' are not. With this in mind, we can introduce some more definitions.

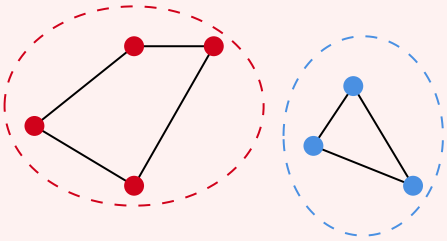
Definition 1.16 (Connectivity)

In a graph $G = (V, E)$, we say that two vertices $u, v \in V$ are **connectied** if there is some path between u and b using only edges in E.

Definition 1.17 (Connected Graph)
If there is a path between any two vertices in G then we say that G is **connected**.

Definition 1.18 (Connected Components)
A **component** of a graph G is a subgraph in which any two vertices in the subgraph are connected, but no vertex in the subgraph is connected to a vertex outside of the subgraph.

Example 1.19
In the graph below, the vertices that are the same colour are in the same component.



§1.2 Trees

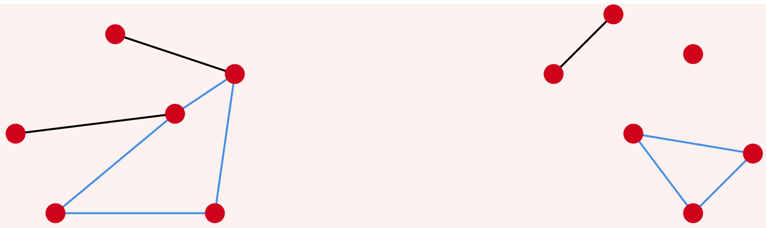
We will now discuss a special class of graph called *trees*. This class is quite restrictive (yet is quite useful), and they have some nice properties.

To define what a tree is, we first need a notion of when a graph is acyclic.

Definition 1.20 (Acyclic)
A graph G is said to be **acyclic** if it does not contain any subgraph isomorphic to a cycle, C_n.

Example 1.21 (Example of Acyclic/Non-Acyclic Graphs)
In the example below, the two graphs are both *acyclic*.

Two *non-acyclic* graphs are shown below. The subgraphs isomorphic to C_4 and C_3 are highlighted.

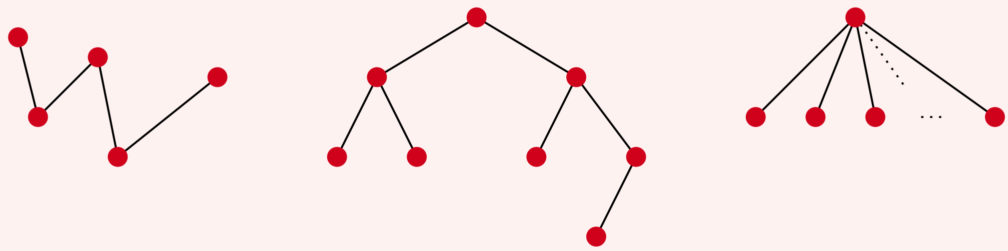


Definition 1.22 (Tree)

A **tree** is a connected, acyclic graph.

Example 1.23 (Examples of Trees)

The following three graphs are trees.



Proposition 1.24 (Characterising Trees)

The following are equivalent.

(a) G is a tree.

(b) G is a maximal acyclic graph (adding any edge creates a cycle).

(c) G is a minimal connected graph (removing any edge disconnects the graph).

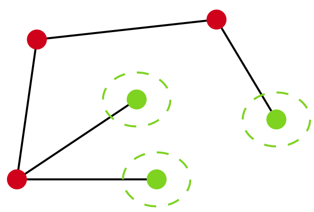
Proof. We will prove that (a) implies (b), and leave the rest as an exercise.

By definition $G = (V, E)$ is acyclic and connected. So let $x, y \in V$ such that $xy \notin E$. As G is connected, there is some path between x and y in the graph, so then adding the edge xy to this path gives us a cycle. □

Definition 1.25 (Leaf)

Let G be a graph. A vertex $v \in V(G)$ is a **leaf** if $d(v) = 1$.

For example, the tree below has three leaves.



In general, trees have a leaves.

Proposition 1.26 (Trees Have Leaves)
Every tree T with $|T| \geq 2$ has a leaf.

Proof. Let P be a path of maximum length in T, with $P = x_1 x_2 \dots x_k$. We claim $\deg(x_k) = 1$. Suppose x_k was adjacent to some vertex y other than x_{k-1}. Then if y already occurred in the path the graph would contain a cycle which is a contradiction, and otherwise if y did not occur in the path, it could be added to it, increasing the length and contradicting maximality. Thus there can be no other vertex adjacent to x_k. □

Remark. This proof gives us two leaves in T, which is the best we can hope for considering P_n is a tree with exactly two leaves.

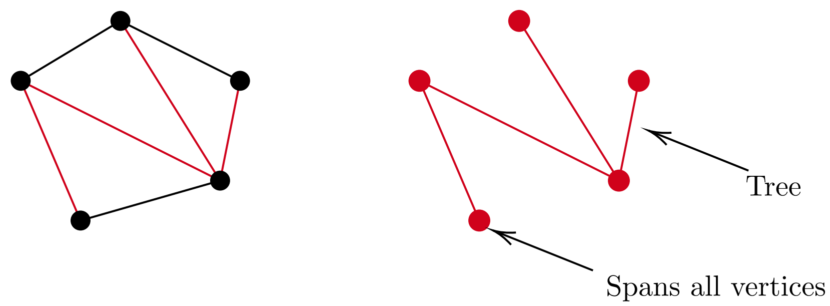
Proposition 1.27 (Edges of a Tree)
Let T be a tree. Then $e(T) = |T| - 1$.

Proof. Left as an exercise[a]. □

[a] If you want a hint, try use induction.

Now lets think about trees as subgraphs of other graphs.

Definition 1.28 (Spanning Tree)
Let $G = (V, E)$ be a graph. We say T is a **spanning tree** of G if T is a tree on V and is a subgraph of G.



Spanning trees are useful in lots of contexts, one of which is giving a sensible ordering to the vertices of a graph. They are particularly useful because of the following result.

Proposition 1.29 (Connected Graphs have Spanning Trees)
Every connected graph contains a spanning tree.

Proof. A tree is a minimal connected graph. So take the connected graph and remove edges until it becomes a minimal connected graph. Then this will be a subgraph of the original graph, and will thus be a spanning tree. □

§1.3 Problem Solving in Graph Theory

So far we haven't really proved many theorems about graphs, but we will soon change that as we look at some problems. Before we do that though, it's worth talking about some common strategies that are used in graph theory problems.

One strategy you should always have to hand is using *induction*. Induction works unusually well for proving facts in graph theoretic settings, though it's not always obvious how it should be applied. Usually inductive proofs in graph theory look like:

1. Suppose that the result you care about hold for $n-1$.
2. Take a graph with n. Remove things so that it has $n-1$, then apply the inductive hypothesis.
3. Add back on what you removed, and show that either everything still works or that you can do something to make it work.

Here's some concrete examples.

Example 1.30 (Veblen's Theorem)

Prove that the edges of a graph can be partitioned into cycles if and only if each vertex has even degree.

Walkthrough.

(a) Suppose the graph can be partitioned into cycles, then show that every vertex has even degree.

(b) Prove that a cycle exists.

(c) Remove it from the graph and apply the inductive hypothesis.

Example 1.31 (Euler's Formula)

Let G be a connected planar[a] graph with V vertices, E edges and F faces. Then

$$V - E + F = 2.$$

[a] We say that a graph is **planar** if it can be drawn on the plane $\mathbb{R}^2$ that no edges overlap. We will also call the connected components of $\mathbb{R}^2 - G$ the **faces** of the graph.

Walkthrough.

(a) Prove it for when G is a tree.

(b) Suppose the graph contained a cycle. Remove it, apply the inductive hypothesis.

Of course, as with most problems in Combinatorics, you will need to be creative (which is what makes the problems fun!)

§1.4 Problems

There are lots of problems here! Since you most likely have limited time, I gave each problem a points value, indicating how much I like it and how useful I think it would be to do. They *do not correspond with difficulty* and you should try the problems with the highest points that you like the look of.

> ***Instructions***: Try to solve at least [10♣] during the break, and make sure you at least try the mandatory problems (which are in red).

Problem 1 (3♣, Handshaking Lemma)**.** Prove that if $G = (V, E)$ is a graph then

$$\sum_{v \in V} \deg v = 2E.$$

Problem 2 (1♣)**.** Is it possible to build a house with exactly eight rooms, each with three doors, and such that exactly three of the house's doors lead outside?

Problem 3 (1♣)**.** How many graphs are there with at most 5 vertices?

Problem 4 (3♣)**.** Let G be a disconnected graph. Prove that it's complement[2] $\overline{G}$ is connected.

Problem 5 (3♣)**.** Let G be a graph with at least as many edges as vertices. Show it has some cycle.

Problem 6 (4♣)**.** Show that at any party, there are always at least two people with exactly the same number of friends at the party.

Problem 7 (5♣)**.** Define a k-clique to be a set of k people such that every pair of them are acquainted with each other. At a certain party, every pair of 3-cliques has at least one person in common, and there are no 5-cliques. Prove that there are two or fewer people at the party whose departure leaves no 3-clique remaining.

Problem 8 (4♣)**.** My wife and I were invited to a dinner party attended by four other couples, making a total of 10 people. A certain amount of handshaking took place subject to two conditions: no one shook his or her own hand and no couple shook hands with each other. Afterwards, I became curious and asked everybody else at the party how many people they shook hands with. Given that I received nine different answers, how many hands did I shake?

Problem 9 (5♣)**.** Prove that in a party with 6 people, there must exist three mutual friends or three mutual strangers. Show that this is not true for a party with 5 people

Problem 10 (9♣, IrMO 1989 P7)**.** Each of the n members of a club is given a different item of information. They are allowed to share the information, but, for security reasons, only in the following way: A pair may communicate by telephone. During a telephone call only one member may speak. The member who speaks may tell the other member all the information s(he) knows. Determine the minimal number of phone calls that are required to convey all the information to each other.

[2] The **complement** of a graph is the graph obtained by including an edge if and only if it was not present in the original graph.

Problem 11 (5♣, IrMO 1994 P10)**.** If a square is partitioned into n convex polygons, determine the maximum number of edges present in the resulting figure.

Problem 12 (9♣)**.** A social network has 2019 users, some pairs of whom are friends. Whenever user A is friends with user B, user B is also friends with user A. Events of the following kind may happen repeatedly, one at a time:

> Three users A, B, and C such that A is friends with both B and C, but B and C are not friends, change their friendship statuses such that B and C are now friends, but A is no longer friends with B, and no longer friends with C. All other friendship statuses are unchanged.

Initially, 1010 users have 1009 friends each, and 1009 users have 1010 friends each. Prove that there exists a sequence of such events after which each user is friends with at most one other user.