Inversion*

Olympiad Training Notes
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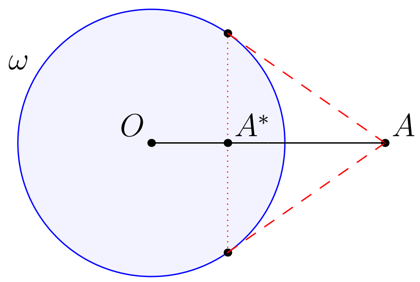
Rules of the Game

Let ω be a circle with center O and radius r. An *inversion* about ω is a transformation satisfying the following.

- The center O of the circle is sent to P_∞.
- The point P_∞ is sent to O.
- Any other point A is sent to the point A^* on the ray OA such that $OA \cdot OA^* = r^2$.

A Geometric Interpretation

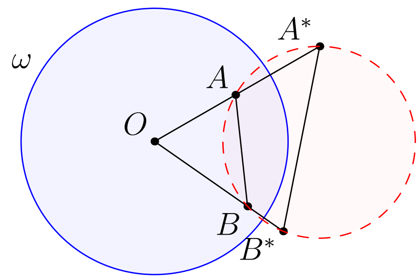
First notice that inversion swaps points. Then we can use the following construction find the inverse of points with respect to ω.



Proof Sketch. Check $OA \cdot OA^* = r^2$. □

Inverting Pairs of Points

Consider points A and B and circle ω.



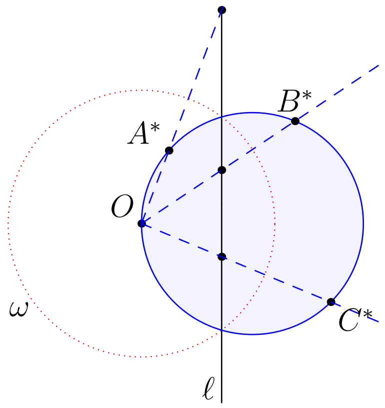
Inverting about ω, notice that $OA \cdot OA^* = OB \cdot OB^* = r^2$ implies AA^*BB^* are concyclic by power of a point. Notably that gives us the angle condition $\measuredangle OAB = -\measuredangle OB^*A^*$.

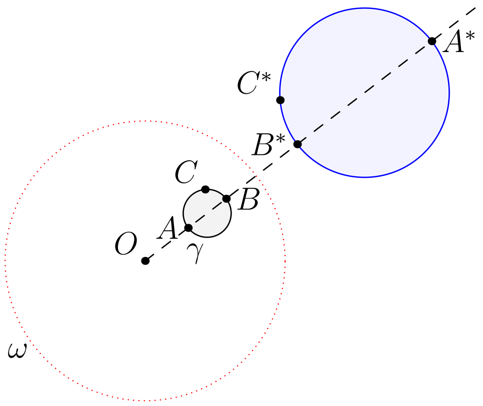
*Based on chapter 8 of Evan Chen's 'Euclidean Geometry in Mathematical Olympiads'

Inverting Circles and Lines

In an inversion about a circle with center O,

- A line through O inverts to itself.
- A circle through O inverts to a line not through O, and vice versa. The diameter of this circle containing O is perpendicular to the line.
- A circle not through O inverts to another circle not through O. The centers of these circles are collinear with O.





Proof Sketch. Angle chase using the angle condition we obtained in the 'Inverting Pairs of Points' section. □

Inversion and Distance

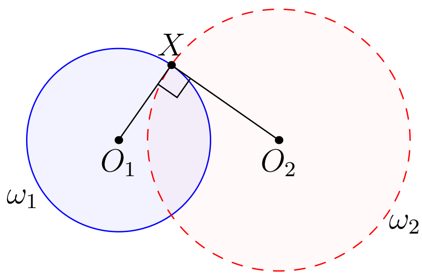
Let A and B be points other than O and consider an inversion about O with radius r. Then

$$A^*B^* = \frac{r^2}{OA \cdot OB} \cdot AB, \qquad \text{and} \qquad AB = \frac{r^2}{OA^* \cdot OB^*} \cdot A^*B^*.$$

Proof Sketch. Use similar triangles, and the second formula follows directly. □

Inverting Orthogonal Circles

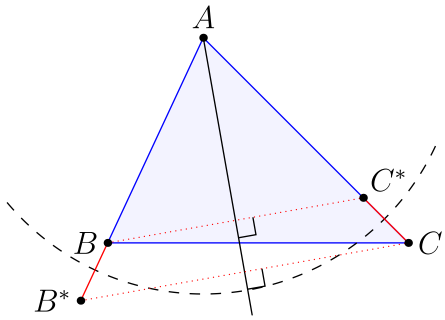
In the diagram below, we call the circles ω_1 and ω_2 *orthogonal*. Inverting about ω_1, ω_2 inverts to itself (and vice versa).



Proof Sketch. Use power of a point with respect to ω_2. □

Forcing a Swap – $\sqrt{bc}$ Inversion

Inversion about a circle at A with radius $\sqrt{AB \cdot AC}$, followed by a reflection across the bisector of $\angle BAC$ swaps B and C.



Proof Sketch. Computation. □