IRMO ALGEBRA

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Remark. This is a collection of all algebra problems that have appeared in the Irish Mathematical Olympiad and the Irish EGMO selection test. The questions are ordered chronologically. All problems are due to their respective creators.

EGMO Selection Test Problems

Problem 1 (EGMO TST 2019)**.** For any positive integer $n \geq 1$ denote $n! = 1 \cdot 2 \cdot 3 \cdots n$. Prove that:

(a) for any integer $n \geq 1$ we have

$$\frac{(2n)^2}{(2n-1)!(2n+1)!} < \frac{1}{(2n-1)!} - \frac{1}{(2n+1)!}$$

(b) We have
$$\frac{2^2}{1!3!} + \frac{4^2}{3!5!} + \frac{6^2}{5!7!} + \cdots + \frac{2018^2}{2017!2019!} < 1 - \frac{1}{2019!}$$

Problem 2 (EGMO TST 2019)**.** Finn has 5 distinct real numbers. He takes the sum of each pair of numbers and writes down the 10 sums. The 3 smallest sums are $30, 34$ and 35, while the 2 largest are 46 and 49. Determine, with proof, the largest of Finn's 5 numbers.

Problem 3 (EGMO TST 2018)**.** The non-zero real numbers a, b, c, d satisfy the equalities
$$a + b + c + d = 0, \quad \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \frac{1}{abcd} = 0$$
Find, with proof, all possible values of the product $(ab - cd)(c + d)$

Problem 4 (EGMO TST 2018)**.** Let $\{S_n : n = 0, 1, 2 \ldots\}$ be a sequence defined by $S_0 = 1$ and $S_n = S_{n-1} + \frac{1}{\sqrt{n}}$ for $n \geq 1$ Show that $S_n \leq 2\sqrt{n}$, for all $n \geq 1$

Problem 5 (EGMO TST 2015)**.** (a) Which is the larger number: $A = 200!$ or $B = 100^{200}$? Justify your answer.

(b) Which is the larger number: $A = 2000!$ or $B = 100^{2000}$? Justify your answer.

Problem 6 (EGMO TST 2015)**.** For any positive integer k define

$$H_k = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k}$$

Prove that for $n \geq 1$

$$1 + \frac{1}{n+1}(H_1 + H_2 + \cdots + H_n) = H_{n+1}$$

Problem 7 (EGMO TST 2014)**.** Let j, n be two integers such that $n \geq 1$ and $0 \leq j \leq n$. Prove that
$$\sum_{k=j}^{n} \binom{n}{k} \binom{k}{j} = 2^{n-j} \binom{n}{j}$$

Problem 8 (EGMO TST 2013)**.** Let $n \geq 1$ be a positive integer. Evaluate in terms of n the sum

$$1 \cdot 2 \cdot 4 + 2 \cdot 3 \cdot 5 + 3 \cdot 4 \cdot 6 + ... + n(n+1)(n+3)$$

Problem 9 (EGMO TST 2013)**.** Determine with proof the largest and smallest of the three numbers

$$\sqrt{7}, 1 + \sqrt[3]{3}, \sqrt[4]{67}$$

IrMO Problems

Problem 1 (IrMO 2020 Q5)**.** Let $a, b, c > 0$. Prove that

$$\sqrt[7]{\frac{a}{b+c}+\frac{b}{c+a}}+\sqrt[7]{\frac{b}{c+a}+\frac{c}{a+b}}+\sqrt[7]{\frac{c}{a+b}+\frac{a}{b+c}} \geq 3.$$

Problem 2 (IrMO 2019 Q4)**.** Find the set of all quadruplets (x, y, z, w) of non-zero real numbers which satisfy

$$1+\frac{1}{x}+\frac{2(x+1)}{xy}+\frac{3(x+1)(y+2)}{xyz}+\frac{4(x+1)(y+2)(z+3)}{xyzw}=0$$

Problem 3 (IrMO 2019 Q7)**.** Three non-zero real numbers a, b, c satisfy $a+b+c=0$ and $a^4+b^4+c^4=128$ Determine all possible values of $ab+bc+ca$

Problem 4 (IrMO 2018 Q9)**.** The sequence of positive integers $a_1, a_2, a_3, \ldots$ satisfies

$$a_{n+1}=a_n^2+2018 \quad \text{for } n \geq 1$$

Prove that there exists at most one n for which a_n is the cube of an integer.

Problem 5 (IrMO 2017 Q2)**.** Solve the equations

$$a+b+c=0, \quad a^2+b^2+c^2=1, \quad a^3+b^3+c^3=4abc$$

for a, b, and c

Problem 6 (IrMO 2017 Q5)**.** The sequence $a=(a_0, a_1, a_2, \ldots)$ is defined by $a_0=0, a_1=2$ and

$$a_{n+2}=2a_{n+1}+41a_n \text{ for all } n \geq 0$$

Prove that a_{2016} is divisible by 2017 .

Problem 7 (IrMO 2017 Q10)**.** Given a positive integer m, a sequence of real numbers $a=(a_1, a_2, a_3, \ldots)$ is called m-powerful if it satisfies $\left(\sum_{k=1}^n a_k\right)^m=\sum_{k=1}^n a_k^m$ for all positive integers n

(a) Show that a sequence is 30 -powerful if and only if at most one of its terms is non-zero.

(b) Find a sequence none of whose terms is zero but which is 2017 -powerful.

Problem 8 (IrMO 2016 Q3)**.** Do there exist four polynomials $P_1(x), P_2(x), P_3(x), P_4(x)$ with real coefficients, such that the sum of any three of them always has a real root, but the sum of any two of them has no real root?

Problem 9 (IrMO 2016 Q8)**.** Suppose a, b, c are real numbers such that $abc \neq 0$. Determine x, y, z in terms of a, b, c such that

$$bz+cy=a, cx+az=b, ay+bx=c$$

Prove also that

$$\frac{1x^2}{a^2}=\frac{1y^2}{b^2}=\frac{1z^2}{c^2}$$

Problem 10 (IrMO 2016 Q9)**.** Show that the number

$$\left(\frac{251}{\frac{1}{\sqrt[3]{252}-5\sqrt[3]{2}}-10\sqrt[3]{63}}+\frac{1}{\frac{251}{\sqrt[3]{252}+5\sqrt[3]{2}}+10\sqrt[3]{63}}\right)^3$$

is an integer and find its value.

Problem 11 (IrMO 2015 Q5)**.** Suppose a doubly infinite sequence of real numbers

$$a_{-2}, a_{-1}, a_0, a_1, a_2, \dots$$

has the property that

$$a_{n+3} = \frac{a_n + a_{n+1} + a_{n+2}}{3},$$

for all integers n.

Show that if this sequence is bounded (i.e., if there exists a number R such that $|a_n| \leq R$ for all n) , then a_n has the same value for all n.

Problem 12 (IrMO 2015 Q9)**.** Let $p(x)$ and $q(x)$ be non-constant polynomial functions with integer coefficients. It is known that the polynomial

$$p(x)q(x) - 2015$$

has at least 33 different integer roots. Prove that neither $p(x)$ nor $q(x)$ can be a polynomial of degree less than three.

Problem 13 (IrMO 2014 Q4)**.** Three different nonzero real numbers a, b, c satisfy the equations

$$a + \frac{2}{b} = b + \frac{2}{c} = c + \frac{2}{a} = p$$

where p is a real number. Prove that $abc + 2p = 0$

Problem 14 (IrMO 2014 Q8)**.** (a) Let a_0, a_1, a_2 be real numbers and consider the polynomial $P(x) = a_0 + a_1x + a_2x^2$. Assume that $P(-1), P(0)$ and $P(1)$ are integers. Prove that $P(n)$ is an integer for all integers n

(b) Let a_0, a_1, a_2, a_3 be real numbers and consider the polynomial $Q(x) = a_0 + a_1x + a_2x^2 + a_3x^3$. Assume that there exists an integer i such that $Q(i)$ $Q(i+1), Q(i+2)$ and $Q(i+3)$ are integers. Prove that $Q(n)$ is an integer for all integers n

Problem 15 (IrMO 2014 Q9)**.** Let n be a positive integer and $a_1, \dots, a_n$ be positive real numbers. Let $g(x)$ denote the product

$$(x + a_1) \cdots (x + a_n)$$

Let a_0 be a real number and let

$$f(x) = (x - a_0)\, g(x) = x^{n+1} + b_1x^n + b_2x^{n-1} + \dots + b_nx + b_{n+1}$$

Prove that all the coefficients $b_1, b_2, \dots, b_{n+1}$ of the polynomial $f(x)$ are negative if and only if

$$a_0 > a_1 + a_2 + \dots + a_n$$

Problem 16 (IrMO 2013 Q9)**.** We say that the doubly infinite sequence

$$\dots, s_{-2}, s_{-1}, s_0, s_1, s_2, \dots$$

is *subaveraging* if $s_n = (s_{n-1} + s_{n+1})/4$ for all integers n.

(a) Find a subaveraging sequence in which all entries are different from each other. Prove that all entries are indeed distinct.

(b) Show that if (s_n) is a subaveraging sequence such there exist distinct integers m, n such that $s_m = s_n$, then there are infinitely many pairs of distinct integers i, j with $s_i = s_j$.

Problem 17 (IrMO 2013 Q10)**.** Let a, b, c be real numbers and let $x = a + b + c, y = a^2 + b^2 + c^2, z = a^3 + b^3 + c^3$ and $S = 2x^3 - 9xy + 9z$

(a) Prove that S is unchanged when a, b, c are replaced by $a+t, b+t, c+t$ respectively, for any real number t

(b) Prove that $\left(3y - x^2\right)^3 \geq 2S^2$

Problem 18 (IrMO 2011 Q1)**.** Suppose $abc \neq 0$. Express in terms of a, b, and c the solutions x, y, z, u, v, w of the equations

$$x + y = a, \quad z + u = b, \quad v + w = c, \quad ay = bz, \quad ub = cv, \quad wc = ax$$

Problem 19 (IrMO 2011 Q3)**.** The integers $a_0, a_1, a_2, a_3, \ldots$ are defined as follows:

$$a_0 = 1, \quad a_1 = 3, \quad \text{ and } a_{n+1} = a_n + a_{n-1} \text{ for all } n \geq 1$$

Find all integers $n \geq 1$ for which $na_{n+1} + a_n$ and $na_n + a_{n-1}$ share a common factor greater than 1

Problem 20 (IrMO 2010 Q7)**.** For each odd integer $p \geq 3$ find the number of real roots of the polynomial

$$f_p(x) = (x-1)(x-2)\cdots(x-p+1)+1$$

Problem 21 (IrMO 2010 Q9)**.** Let $n \geq 3$ be an integer and $a_1, a_2, \ldots, a_n$ be a finite sequence of positive integers, such that, for $k = 2, 3, \ldots, n$

$$n\left(a_k + 1\right) - (n-1)a_{k-1} = 1$$

Prove that a_n is not divisible by $(n-1)^2$

Problem 22 (IrMO 2009 Q6)**.** Let $p(x)$ be a polynomial with rational coefficients. Prove that there exists a positive integer n such that the polynomial $q(x)$ defined by

$$q(x) = p(x+n) - p(x)$$

has integer coefficients.

Problem 23 (IrMO 2008 Q2)**.** For positive real numbers a, b, c and d such that $a^2 + b^2 + c^2 + d^2 = 1$ prove that

$$a^2b^2cd + ab^2c^2d + abc^2d^2 + a^2bcd^2 + a^2bc^2d + ab^2cd^2 \leq \frac{3}{32}$$

and determine the cases of equality.

Problem 24 (IrMO 2008 Q8)**.** Find $a_3, a_4, \ldots, a_{2008}$, such that $a_i = \pm 1$ for $i = 3, \ldots, 2008$ and

$$\sum_{i=3}^{2008} a_i 2^i = 2008$$

and show that the numbers $a_3, a_4, \ldots, a_{2008}$ are uniquely determined by these conditions.

Problem 25 (IrMO 2007 Q6)**.** Let r, s and t be the roots of the cubic polynomial

$$p(x) = x^3 - 2007x + 2002$$

Determine the value of

$$\frac{r-1}{r+1} + \frac{s-1}{s+1} + \frac{t-1}{t+1}$$

Problem 26 (IrMO 2007 Q10)**.** Suppose a and b are real numbers such that the quadratic polynomial

$$f(x) = x^2 + ax + b$$

has no nonnegative real roots. Prove that there exist two polynomials g, h, whose coefficients are nonnegative real numbers, such that

$$f(x) = \frac{g(x)}{h(x)}$$

for all real numbers x.

Problem 27 (IrMO 2006 Q4)**.** Find the greatest value and the least value of $x + y$, where x and y are real numbers, with $x \geq -2, y \geq -3$ and

$$x - 2\sqrt{x+2} = 2\sqrt{y+3} - y$$

Problem 28 (IrMO 2004 Q4)**.** Prove that there are only two real numbers x such that

$$(x-1)(x-2)(x-3)(x-4)(x-5)(x-6) = 720$$

Problem 29 (IrMO 2004 Q4)**.** Define the function m of the three real variables x, y, z by

$$m(x, y, z) = \max\left(x^2, y^2, z^2\right), x, y, z \in \mathbb{R}$$

Determine, with proof, the minimum value of m if x, y, z vary in $\mathbb{R}$ subject to the following restrictions:

$$x + y + z = 0, \quad x^2 + y^2 + z^2 = 1$$

Problem 30 (IrMO 2003 Q3)**.** For each positive integer k, let a_k be the greatest integer not exceeding $\sqrt{k}$ and let b_k be the greatest integer not exceeding $\sqrt[3]{k}$. Calculate

$$\sum_{k=1}^{2003} (a_k - b_k)$$

Problem 31 (IrMO 2003 Q9)**.** Let $a, b > 0$. Determine the largest number c such that

$$c \leq \max\left(ax + \frac{1}{ax}, bx + \frac{1}{bx}\right)$$

for all $x > 0$

Problem 32 (IrMO 2002 Q9)**.** For each real number x, define $\lfloor x \rfloor$ to be the greatest integer less than or equal to x Let $\alpha = 2 + \sqrt{3}$. Prove that

$$\alpha^n - \lfloor \alpha^n \rfloor = 1 - \alpha^{-n}, \text{ for } n = 0, 1, 2, \ldots$$

Problem 33 (IrMO 2000 Q5)**.** Let $p(x) = a_0 + a_1x + \cdots + a_nx^n$ be a polynomial with non-negative real coefficients. Suppose that $p(4) = 2$ and that $p(16) = 8$. Prove that $p(8) \leq 4$ and find, with proof, all such polynomials with $p(8) = 4$.

Problem 34 (IrMO 1999 Q5)**.** Three real numbers a, b, c with $a < b < c$, are said to be in arithmetic progression if $c - b = b - a$ Define a sequence $u_n, n = 0, 1, 2, 3, \ldots$ as follows: $u_0 = 0, u_1 = 1$ and, for each $n \geq 1, u_{n+1}$ is the smallest positive integer such that $u_{n+1} > u_n$ and $\{u_0, u_1, \ldots, u_n, u_{n+1}\}$ contains no three elements that are in arithmetic progression. Find u_{100}

Problem 35 (IrMO 1999 Q6)**.** Solve the system of (simultaneous) equations

$$y^2 = (x+8)\left(x^2+2\right)$$
$$y^2 = (8+4x)y + 5x^2 - 16x - 16$$

Problem 36 (IrMO 1998 Q9)**.** A sequence of real numbers x_n is defined recursively as follows: x_0, x_1 are arbitrary positive real numbers, and

$$x_{n+2} = \frac{1 + x_{n+1}}{x_n}, n = 0, 1, 2, \ldots$$

Find x_{1998}

Problem 37 (IrMO 1995 Q7)**.** Suppose that a, b and c are complex numbers, and that all three roots z of the equation

$$x^3 + ax^2 + bx + c = 0$$

satisfy $|z| = 1$ (where $||$ denotes absolute value). Prove that all three roots w of the equation

$$x^3 + |a|x^2 + |b|x + |c| = 0$$

also satisfy $|w| = 1$

Problem 38 (IrMO 1994 Q1)**.** A sequence x_n is defined by the rules: $x_1 = 2$ and

$$nx_n = 2(2n-1)x_{n-1}, \quad n = 2, 3, \ldots$$

Prove that x_n is an integer for every positive integer n

Problem 39 (IrMO 1994 Q7)**.** Let p, q, r be distinct real numbers that satisfy the equations

$$q = p(4-p)$$
$$r = q(4-q)$$
$$p = r(4-r)$$

Find all possible values of $p + q + r$

Problem 40 (IrMO 1994 Q9)**.** Let w, a, b and c be distinct real numbers with the property that there exist real numbers x, y and z for which the following equations hold:

$$x + y + z = 1$$
$$xa^2 + yb^2 + zc^2 = w^2$$
$$xa^3 + yb^3 + zc^3 = w^3$$
$$xa^4 + yb^4 + zc^4 = w^4$$

Express w in terms of a, b and c.

Problem 41 (IrMO 1993 Q1)**.** The real numbers α, β satisfy the equations

$$\alpha^3 - 3\alpha^2 + 5\alpha - 17 = 0$$
$$\beta^3 - 3\beta^2 + 5\beta + 11 = 0$$

Find $\alpha + \beta$

Problem 42 (IrMO 1993 Q4)**.** Let $a_0, a_1, \ldots, a_{n-1}$ be real numbers, where $n \geq 1$, and let the polynomial

$$f(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_0$$

be such that $|f(0)| = f(1)$ and each root α of f is real and satisfies $0 < \alpha < 1$ Prove that the product of the roots does not exceed $1/2^n$.

Problem 43 (IrMO 1993 Q5)**.** Given a complex number $z = x+iy(x, y \text{ real})$, we denote by $P(z)$ the corresponding point (x, y) in the plane. Suppose $z_1, z_2, z_3, z_4, z_5, \alpha$ are nonzero complex numbers such that

(a) $P(z_1), P(z_2), P(z_3), P(z_4), P(z_5)$ are the vertices of a convex pentagon **Q** containing the origin 0 in its interior and

(b) $P(\alpha z_1), P(\alpha z_2), P(\alpha z_3), P(\alpha z_4)$ and $P(\alpha z_5)$ are all inside **Q**

If $\alpha = p + iq$, where p and q are real, prove that $p^2 + q^2 \leq 1$ and that

$$p + q\tan(\pi/5) \leq 1$$

Problem 44 (IrMO 1993 Q7)**.** Let $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$ be $2n$ real numbers, where $a_1, a_2, \ldots, a_n$ are distinct, and suppose that there exists a real number α such that the product

$$(a_i + b_1)(a_i + b_2)\ldots(a_i + b_n)$$

has the value α for $i = 1, 2, \ldots, n$. Prove that there exists a real number β such that the product

$$(a_1 + b_j)(a_2 + b_j)\ldots(a_n + b_j)$$

has the value β for $j = 1, 2, \ldots, n$

Problem 45 (IrMO 1993 Q9)**.** Let x be a real number with $0 < x < \pi$. Prove that, for all natural numbers n, the

$$\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \ldots + \frac{\sin(2n-1)x}{2n-1}$$

is positive.

Problem 46 (IrMO 1992 Q1)**.** Describe in geometric terms the set of points (x, y) in the plane such that x and y satisfy the condition $t^2 + yt + x \geq 0$ for all t with $-1 \leq t \leq 1$

Problem 47 (IrMO 1992 Q2)**.** How many ordered triples (x, y, z) of real numbers satisfy the system of equations

$$x^2 + y^2 + z^2 = 9$$
$$x^4 + y^4 + z^4 = 33$$
$$xyz = -4?$$

Problem 48 (IrMO 1992 Q8)**.** Let a, b, c and d be real numbers with $a \neq 0$. Prove that if all the roots of the cubic equation

$$az^3 + bz^2 + cz + d = 0$$

lie to the left of the imaginary axis in the complex plane, then

$$ab > 0, bc - ad > 0, ad > 0$$

Problem 49 (IrMO 1991 Q5)**.** Find all polynomials

$$f(x) = x^n + a_1 x^{n-1} + \cdots + a_n$$

with the following properties:

- all the coefficients $a_1, a_2, \ldots, a_n$ belong to the set $\{-1, 1\}$
- all the roots are real.

Problem 50 (IrMO 1990 Q8)**.** Let t be a real number, and let

$$a_n = 2\cos\left(\frac{t}{2^n}\right) - 1, \quad n = 1, 2, 3, \ldots$$

Let b_n be the product $a_1 a_2 a_3 \cdots a_n$. Find a formula for b_n that does not involve a product of n terms, and deduce that

$$\lim_{n\to\infty} b_n = \frac{2\cos t + 1}{3}$$

Problem 51 (IrMO 1988 Q7)**.** A function f, defined on the set of real numbers $\mathbb{R}$ is said to have a horizontal chord of length $a > 0$ if there is a real number x such that $f(a + x) = f(x)$. Show that the cubic

$$f(x) = x^3 - x$$

(where $x \in \mathbb{R}$) has a horizontal chord of length a if, and only if, $0 < a \leq 2$

Problem 52 (IrMO 1988 Q11)**.** If facilities for division are not available, it is sometimes convenient in determining the decimal expansion of $1/a, a > 0$, to use the iteration

$$x_{k+1} = x_k\left(2 - a x_k\right), \quad k = 0, 1, 2, \ldots$$

where x_0 is a selected "starting" value. Find the limitations, if any, on the starting values x_0, in order that the above iteration converges to the desired value $1/a$

Problem 53 (IrMO 1988 Q12)**.** Prove that if n is a positive integer, then

$$\sum_{k=1}^{n} \cos^4\left(\frac{k\pi}{2n+1}\right) = \frac{6n-5}{16}$$