IRMO FUNCTIONAL EQUATIONS

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Remark. This is a collection of all functional equation problems that have appeared in the Irish Mathematical Olympiad. The questions are ordered chronologically. All problems are due to their respective creators.

IrMO Problems

Problem 1 (IrMO 2018)**.** Find all functions $f(x) = ax^2 + bx + c$, with $a \neq 0$, such that

$$f(f(1)) = f(f(0)) = f(f(-1))$$

Problem 2 (IrMO 2018)**.** Find all real-valued functions f satisfying

$$f(2x + f(y)) + f(f(y)) = 4x + 8y$$

for all real numbers x and y

Problem 3 (IrMO 2012)**.** Find, with proof, all polynomials f such that f has nonnegative integer coefficients, $f(1) = 8$ and $f(2) = 2012$.

Problem 4 (IrMO 2010)**.** Find all polynomials $f(x) = x^3 + bx^2 + cx + d$, where b, c, d are real numbers, such that $f\left(x^2 - 2\right) = -f(-x)f(x)$.

Problem 5 (IrMO 2006)**.** Determine, with proof, all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(1) = 1$, and

$$f(xy + f(x)) = xf(y) + f(x)$$

for all $x, y \in \mathbb{R}$

Problem 6 (IrMO 2003)**.** Show that there is no function f defined on the set of positive real numbers such that

$$f(y) > (y - x)(f(x))^2$$

for all x, y with $y > x > 0$.

Problem 7 (IrMO 2002)**.** Denote by $\mathbb{Q}$ the set of rational numbers. Determine all functions $f : \mathbb{Q} \longrightarrow \mathbb{Q}$ such that

$$f(x + f(y)) = y + f(x), \text{ for all } x, y \in \mathbb{Q}$$

Problem 8 (IrMO 2001)**.** Determine, with proof, all functions f from the set of positive integers to itself which satisfy

$$f(x + f(y)) = f(x) + y$$

for all positive integers x, y.

Problem 9 (IrMO 1999)**.** A function $f : \mathbb{N} \to \mathbb{N}$ (where $\mathbb{N}$ denotes the set of positive integers) satisfies

(a) $f(ab) = f(a)f(b)$ whenever the greatest common divisor of a and b is 1

(b) $f(p + q) = f(p) + f(q)$ for all prime numbers p and q

Prove that $f(2) = 2, f(3) = 3$ and $f(1999) = 1999$

Problem 10 (IrMO 1996)**.** Let K be the set of all real numbers x such that $0 \leq x \leq 1$. Let f be a function from K to the set of all real numbers $\mathbb{R}$ with the following properties

(a) $f(1) = 1$

(b) $f(x) \geq 0$ for all $x \in K$

(c) if x, y and $x + y$ are all in K, then $f(x + y) \geq f(x) + f(y)$

Prove that $f(x) \leq 2x$, for all $x \in K$

Problem 11 (IrMO 1995)**.** Determine, with proof, all real-valued functions f satisfying the equation

$$xf(x) - yf(y) = (x - y)f(x + y)$$

for all real numbers x, y.

Problem 12 (IrMO 1994)**.** Determine, with proof, all real polynomials f satisfying the equation

$$f\left(x^2\right) = f(x)f(x - 1)$$

for all real numbers x

Problem 13 (IrMO 1994)**.** Let $f(n)$ be defined on the set of positive integers by the rules: $f(1) = 2$ and

$$f(n + 1) = (f(n))^2 - f(n) + 1, \quad n = 1, 2, 3, ...$$

Prove that, for all integers $n > 1$

$$1 - \frac{1}{2^{2n-1}} < \frac{1}{f(1)} + \frac{1}{f(2)} + ... + \frac{1}{f(n)} < 1 - \frac{1}{2^{2^n}}$$

Problem 14 (IrMO 1991)**.** Let $\mathbb{P}$ be the set of positive rational numbers and let $f : \mathbb{P} \to \mathbb{P}$ be such that

$$f(x) + f\left(\frac{1}{x}\right) = 1$$

and

$$f(2x) = 2f(f(x))$$

for all $x \in \mathbb{P}$ Find, with proof, an explicit expression for $f(x)$ for all $x \in \mathbb{P}$

Problem 15 (IrMO 1991)**.** Find all polynomials

$$f(x) = a_0 + a_1x + \cdots + a_nx^n$$

satisfying the equation

$$f\left(x^2\right) = (f(x))^2$$

for all real numbers x.

Problem 16 (IrMO 1990)**.** Determine whether there exists a function $f : \mathbb{N} \to \mathbb{N}$ (where $\mathbb{N}$ is the set of natural numbers) such that $f(n) = f(f(n - 1)) + f(f(n + 1))$ for all natural numbers $n \geq 2$.

Problem 17 (IrMO 1989)**.** A function f is defined on the natural numbers $\mathbb{N}$ and satisfies the following rules: (a) $f(1) = 1$ (b) $f(2n) = f(n)$ and $f(2n + 1) = f(2n) + 1$ for all $n \in \mathbb{N}$ Calculate the maximum value m of the set $\{f(n) : n \in \mathbb{N}, 1 \leq n \leq 1989\}$, and determine the number of natural numbers n, with $1 \leq n \leq 1989$, that satisfy the equation $f(n) = m$.

Harder Problems

These problems are taken from various competitions, and are (in general) harder than IrMO problems.

Problem 1. Find all functions $f : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$f(x + yf(x)) = f(xf(y)) - x + f(y + f(x))$$

for all x and y real numbers.

Problem 2. Find all functions $f : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$f(f(x) + y) = f(x^2 - y) + 4f(x)y$$

for all x and y real numbers.

Problem 3 (Romanian TST 2011)**.** Find all functions $f : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$2f(x) = f(x + y) + f(x + 2y),$$

for all $x \in \mathbb{R}_+$ and $y \geq 0$.

Problem 4 (IMO SL 2009)**.** Find all functions $f : \mathbb{R}_+ \to \mathbb{R}_P$ such that

$$f(xf(x + y)) = f(yf(x)) + x^2,$$

for all $x, y \in \mathbb{R}_+$.

Problem 5 (IMO SL 1993)**.** Determine all functions $f : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$f(xf(y)) = yf(x) \text{for all} x, y \in \mathbb{R}_+$$

and as $x \to \infty$, then $f(x) \to 0$.

Problem 6 (IMO 1999)**.** Find all functions $f : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$f(x - f(y)) = f(f(y)) + xf(y) + f(x) - 1$$

for all $x, y \in \mathbb{R}_+$.

Problem 7 (IMO 1999)**.** Find all functions $f : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$f(x + f(x + y)) + f(xy) = x + f(x + y) + yf(x)$$

for all $x, y \in \mathbb{R}_+$.