IRMO GEOMETRY

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Remark. This is a collection of all geometry problems that have appeared in the Irish Mathematical Olympiad and the Irish EGMO selection test. The questions are ordered chronologically. All problems are due to their respective creators.

EGMO Selection Test Problems

Problem 1 (EGMO TST 2020)**.** A, B and C are three points on a circle $\mathcal{C}$. Let BD be the bisector of angle $\angle ABC$ and $DE \| AB$, where D, E lie also on $\mathcal{C}$. Prove that $|BC| = |DE|$

Problem 2 (EGMO TST 2020)**.** Let $ABCD$ be a convex quadrilateral. Let M, N, O, P be the midpoints of AB, BC, CD and DA respectively.

(a) Show that $MNOP$ is a parallelogram.

(b) Show that the area $[MNOP]$ is 1/2 of the area $[ABCD]$

Problem 3 (EGMO TST 2020)**.** Let ABC be a triangle. Let $\mathcal{C}_1$ be the circle which passes through A and B and which is tangent to BC. Let $\mathcal{C}_2$ be the circle which passes through A and C and which is tangent to BC. Let T be the point of intersection (other than A) of $\mathcal{C}_1$ and $\mathcal{C}_2$. Show that if $\angle BAT = \angle CAT$ then $|BT| = |CT|$.

Problem 4 (EGMO TST 2019)**.** Let $ABCD$ be a convex quadrilateral. Suppose that $AB = CD$. Prove that $BC \cdot (\sin \angle B - \sin \angle C) = AD \cdot (\sin \angle A - \sin \angle D)$

Problem 5 (EGMO TST 2019)**.** Let ABC be a right-angled triangle with hypotenuse AB. Let D lie on the segment BC with $BD = 2 \cdot DC$. Let M be the midpoint of the hypotenuse AB. Determine the ratio AD/DM.

Problem 6 (EGMO TST 2019)**.** Let $ABCD$ be a square. Let P be any point on the circumcircle of $ABCD$ lying on the arc joining A to B (and distinct from A and B). Let M be the point of intersection of DP with the diagonal AC. Let N be the point of intersection of CP with the side AB. Show that MN is parallel to BD.

Problem 7 (EGMO TST 2018)**.** In triangle ABC, P is a point on AB, Q is a point on AC and X is the point of intersection of the line segments PC and QB. The quadrilateral $APXQ$ has area 4. The triangles QXC and PXB have area 5 and 1 respectively. What is the area of the triangle ABC?

Problem 8 (EGMO TST 2018)**.** Let A, B, C, D be four distinct points on a circle (in this order). Let P be the intersection of AD and BC, and let Q be the intersection of AB and CD. Prove that the angle bisectors of $\angle DPC$ and AQD are perpendicular.

Problem 9 (EGMO TST 2017)**.** The diagonals of the convex quadrilateral $ABCD$ of area 1 intersect at O. If $\frac{BO}{DO} = \frac{1}{2}$ and $\frac{AO}{CO} = \frac{3}{4}$, find the area of triangles AOB, BOC, COD and DOA.

Problem 10 (EGMO TST 2017)**.** ABC is an acute triangle. The bisector AL, the altitude BH and the median CM are such that $\angle CAL = \angle ABH = \angle ACM$. Find the angles of triangle ABC.

Problem 11 (EGMO TST 2016)**.** ABC is an acute triangle and D is a point on the segment BC. Two circles $\mathcal{C}_1$ and $\mathcal{C}_2$ passing through B, D and C, D respectively intersect for the second time at P, where P lies inside of triangle ABC. Denote by R the intersection of $\mathcal{C}_1$ and AB and by Q the intersection of $\mathcal{C}_2$ and AC.

Prove that P lies on the circumcircle of triangle QAR.

Problem 12 (EGMO TST 2016)**.** On sides AB, BC and CA of triangle ABC we consider the points M, N and P respectively such that

$$\frac{AM}{MB} = \frac{BN}{NC} = \frac{CP}{PA} = \frac{1}{2}$$

Prove that:

(a) $[AMN] = \frac{1}{9}$

(b) $[MNP] = \frac{1}{3}[ABC]$

Problem 13 (EGMO TST 2015)**.** In triangle ABC we denote by A', B', C' the midpoints of sides BC, CA and AB respectively. We extend AA' beyond A' with $A'M = AA'$. We extend BB' beyond B' with $B'N = BB'$ and extend CC' beyond C' with $C'P = CC'$. Denote by G_1, G_2 and G_3 the centroids of triangles MBC, NAC and PAB. Prove that triangles ABC and $G_1G_2G_3$ have the same area.

Problem 14 (EGMO TST 2015)**.** A triangle has angles of $36°, 72°$ and $72°$. Prove that it has at least one side whose length is not an integer.

Problem 15 (EGMO TST 2014)**.** A convex quadrilateral $ABCD$ is given. On the extended diagonal BD we consider two points D and E such that B lies between D and E, D lies between B and F and $BE = BD = DF$. Prove that the area of the quadrilateral $AECF$ is three times bigger than the area of $ABCD$.

Problem 16 (EGMO TST 2014)**.** The length of each side of a triangle is an integer and is a divisor of the perimeter of the triangle. Prove that the triangle is equilateral.

Problem 17 (EGMO TST 2014)**.** Let G be the centroid of triangle ABC. Denote by G_1, G_2 and G_3 the centroids of triangles ABG BCG and CAG. Prove that $[G_1G_2G_3] = \frac{1}{9}[ABC]$.

Problem 18 (EGMO TST 2013)**.** Let ABC be an isosceles triangle with $|AB| = |AC|$. The bisector of the angle ABC meets the side AC at the point D.

Prove that if the triangle BCD is isosceles, the triangle ABD must also be isosceles.

Problem 19 (EGMO TST 2012)**.** Three circles of radii $1, 2, 3$ and centres at A, B, C are mutually tangent. Find the area of the triangle ABC.

IrMO Problems

Problem 1 (IrMO 2020 Q10)**.** Show that there exists a hexagon $ABCDEF$ in the plane such that the distance between every pair of vertices is an integer.

Problem 2 (IrMO 2020 Q9)**.** A trapezium $ABCD$, in which AB is parallel to DC, is inscribed in a circle of radius R and centre O. The non-parallel sides DA and CB are extended to meet at P while diagonals AC and BD intersect at E. Prove that $|OE| \cdot |OP| = R^2$.

Problem 3 (IrMO 2020 Q3)**.** Circles Ω_1, centre Q, and Ω_2, centre R, touch externally at B. A third circle, Ω_3, which contains Ω_1 and Ω_2, touches Ω_1 and Ω_2 at A and C, respectively. Point C is joined to B and the line BC is extended to meet Ω_3 at D.

Prove that QR and AD intersect on the circumference of Ω_1.

Problem 4 (IrMO 2019 Q3)**.** A quadrilateral $ABCD$ is such that the sides AB and DC are parallel, and $|BC| = |AB| + |CD|$. Prove that the angle bisectors of the angles $\angle ABC$ and $\angle BCD$ intersect at right angles on the side AD.

Problem 5 (IrMO 2019 Q5)**.** Let M be a point on the side BC of triangle ABC and let P and Q denote the circumcentres of triangles ABM and ACM respectively. Let L denote the point of intersection of the extended lines BP and CQ and let K denote the reflection of L through the line PQ

Prove that M, P, Q and K all lie on the same circle.

Problem 6 (IrMO 2019 Q8)**.** Consider a point G in the interior of a parallelogram $ABCD$. A circle Γ through A and G intersects the sides AB and AD for the second time at the points E and F respectively. The line FG extended intersects the side BC at H and the line EG extended intersects the side CD at I. The circumcircle of triangle HGI intersects the circle Γ for the second time at $M \neq G$. Prove that M lies on the diagonal AC.

Problem 7 (IrMO 2018 Q2)**.** The triangle ABC is right-angled at A. Its incentre is I, and H is the foot of the perpendicular from I on AB. The perpendicular from H on BC meets BC at E and it meets the bisector of $\angle ABC$ at D. The perpendicular from A on BC meets BC at F. Prove that $\angle EFD = 45^\circ$.

Problem 8 (IrMO 2018 Q4)**.** We say that a rectangle with side lengths a and b fits inside a rectangle with side lengths c and d if either ($a \leq c$ and $b \leq d$) or ($a \leq d$ and $b \leq c$). For instance, a rectangle with side lengths 1 and 5 fits inside a rectangle with side lengths 6 and 2 . Suppose S is a set of 2019 rectangles, all with integer side lengths between 1 and 2018 inclusive. Show that there are three rectangles A, B, and C in S such that A fits inside B, and B fits inside C.

Problem 9 (IrMO 2018 Q5)**.** Points A, B and P lie on the circumference of a circle Ω_1 such that $\angle APB$ is an obtuse angle. Let Q be the foot of the perpendicular from P on AB. A second circle Ω_2 is drawn with centre P and radius PQ. The tangents from A and B to Ω_2 intersect Ω_1 at F and H respectively. Prove that FH is tangent to Ω_2

Problem 10 (IrMO 2018 Q8)**.** Let M be the midpoint of side BC of an equilateral triangle ABC. The point D is on CA extended such that A is between D and C. The point E is on AB extended such that B is between A and E, and $|MD| = |ME|$. The point F is the intersection of MD and AB. Prove that $\angle BFM = \angle BME$

Problem 11 (IrMO 2017 Q3)**.** Four circles are drawn with the sides of the quadrilateral $ABCD$ as diameters. The two circles passing through A meet again at A', the two circles through B at B', the two circles through C at C' and the two circles through D at D'. Suppose that the points A', B', C' and D' are distinct. Prove that the quadrilateral $A'B'C'D'$ is similar to the quadrilateral $ABCD$.

[Two quadrilaterals are similar if their corresponding angles are equal to each other and their corresponding side lengths are in proportion to each other.]

Problem 12 (IrMO 2017 Q8)**.** A line segment B_0B_n is divided into n equal parts at points $B_1, B_2, \dots, B_{n1}$ and A is a point such that $\angle B_0AB_n$ is a right angle. Prove that

$$\sum_{k=0}^{n} |AB_k|^2 = \sum_{k=0}^{n} |B_0B_k|^2$$

Problem 13 (IrMO 2016 Q2)**.** In triangle ABC we have $|AB| \neq |AC|$. The bisectors of $\angle ABC$ and $\angle ACB$ meet AC and AB at E and F, respectively, and intersect at I. If $|EI| = |FI|$ find the measure of $\angle BAC$.

Problem 14 (IrMO 2016 Q4)**.** Let ABC be a triangle with $|AC| \neq |BC|$. Let P and Q be the intersection points of the line AB with the internal and external angle bisectors at C, so that P is between A and B. Prove that if M is any point on the circle with diameter PQ then $\angle AMP = \angle BMP$

Problem 15 (IrMO 2016 Q6)**.** Triangle ABC has sides $a = |BC| > b = |AC|$. The points K and H on the segment BC satisfy $|CH| = (a + b)/3$ and $|CK| = (a - b)/3$. If G is the centroid of triangle ABC, prove that $\angle KGH = 90^\circ$.

Problem 16 (IrMO 2016 Q10)**.** Let AE be a diameter of the circumcircle of triangle ABC. Join E to the orthocentre, H, of $\triangle ABC$ and extend EH to meet the circle again at D. Prove that the nine point circle of $\triangle ABC$ passes through the midpoint of HD.

[Note. The nine point circle of a triangle is a circle that passes through the midpoints of the sides, the feet of the altitudes and the midpoints of the line segments that join the orthocentre to the vertices.]

Problem 17 (IrMO 2015 Q1)**.** In the triangle ABC, the length of the altitude from A to BC is equal to $1.D$ is the midpoint of AC. What are the possible lengths of BD? Two circles $\mathcal{C}_1$ and $\mathcal{C}_2$, with centres at D and E respectively, touch at B. The circle having DE as diameter intersects the circle $\mathcal{C}_1$ at H and the circle $\mathcal{C}_2$ at K. The points H and K both lie on the same side of the line $DE.HK$ extended in both directions meets the circle $\mathcal{C}_1$ at L and meets the circle $\mathcal{C}_2$ at M. Prove that

(a) $|LH| = |KM|$

(b) the line through B perpendicular to DE bisects HK

Problem 18 (IrMO 2015 Q8)**.** In triangle $\triangle ABC$, the angle $\angle BAC$ is less than 90°. The perpendiculars from C on AB and from B on AC intersect the circumcircle of $\triangle ABC$ again at D and E respectively. If $|DE| = |BC|$, find the measure of the angle $\angle BAC$.

Problem 19 (IrMO 2014 Q3)**.** In the triangle ABC, D is the foot of the altitude from A to BC, and M is the midpoint of the line segment BC. The three angles $\angle BAD, \angle DAM$ and $\angle MAC$ are all equal. Find the angles of the triangle ABC.

Problem 20 (IrMO 2014 Q7)**.** The square $ABCD$ is inscribed in a circle with centre O. Let E be the midpoint of AD. The line CE meets the circle again at F. The lines FB and AD meet at H. Prove $|HD| = 2|AH|$.

Problem 21 (IrMO 2013 Q3)**.** The altitudes of a triangle ABC are used to form the sides of a second triangle $A_1B_1C_1$. The altitudes of $\triangle A_1B_1C_1$ are then used to form the sides of a third triangle $A_2B_2C_2$. Prove that $\triangle A_2B_2C_2$ is similar to $\triangle ABC$.

Problem 22 (IrMO 2013 Q5)**.** A, B and C are points on the circumference of a circle with centre O. Tangents are drawn to the circumcircles of triangles OAB and OAC at P and Q respectively, where P and Q are diametrically opposite O. The two tangents intersect at K The line CA meets the circumcircle of $\triangle OAB$ at A and X. Prove that X lies on the line KO

Problem 23 (IrMO 2013 Q6)**.** The three distinct points B, C, D are collinear with C between B and D. Another point A not on the line BD is such that $|AB| = |AC|$. Prove that $\angle BAC = 36°$ if and only if

$$\frac{1}{|CD|} - \frac{1}{|BD|} = \frac{1}{|CD| + |BD|}$$

Problem 24 (IrMO 2012 Q2)**.** A, B, C and D are four points in that order on the circumference of a circle K AB is perpendicular to BC and BC is perpendicular to $CD.X$ is a point on the circumference of the circle between A and $D.AX$ extended meets CD extended at E and DX extended meets BA extended at F

Prove that the circumcircle of triangle AXF is tangent to the circumcircle of triangle DXE and that the common tangent line passes through the centre of the circle K

Problem 25 (IrMO 2012 Q4)**.** There exists an infinite set of triangles with the following properties:

(a) the lengths of the sides are integers with no common factors, and

(b) one and only one angle is $60°$.

One such triangle has side lengths $5, 7$ and 8. Find two more.

Problem 26 (IrMO 2012 Q7)**.** Consider a triangle ABC with $|AB| \neq |AC|$. The angle bisector of the angle CAB intersects the circumcircle of $\triangle ABC$ at two points A and D. The circle of centre D and radius $|DC|$ intersects the line AC at two points C and B'. The line BB' intersects the circumcircle of $\triangle ABC$ at B and E. Prove that B' is the orthocentre of $\triangle AED$

Problem 27 (IrMO 2011 Q2)**.** Let ABC be a triangle whose side lengths are, as usual, denoted by $a = |BC|$ $b = |CA|, c = |AB|$. Denote by m_a, m_b, m_c, respectively, the lengths of the medians which connect A, B, C, respectively, with the centres of the corresponding opposite sides.

(a) Prove that $2m_a < b + c$. Deduce that $m_a + m_b + m_c < a + b + c$

(b) Give an example of

i. a triangle in which $m_a > \sqrt{bc}$

ii. a triangle in which $ma \leq \sqrt{bc}$

Problem 28 (IrMO 2011 Q4)**.** The incircle $\mathcal{C}_1$ of triangle ABC touches the sides AB and AC at the points D and E, respectively. The incircle $\mathcal{C}_2$ of the triangle ADE touches the sides AB and AC at the points P and Q, and intersects the circle $\mathcal{C}_1$ at the points M and N. Prove that

(a) the centre of the circle $\mathcal{C}_2$ lies on the circle $\mathcal{C}_1$

(b) the four points M, N, P, Q in appropriate order form a rectangle if and only if twice the radius of $\mathcal{C}_1$ is three times the radius of $\mathcal{C}_2$

Problem 29 (IrMO 2011 Q8)**.** $ABCD$ is a rectangle. E is a point on AB between A and B, and F is a point on AD between A and D. The area of the triangle EBC is 16, the area of the triangle EAF is 12 and the area of the triangle FDC is 30. Find the area of the triangle EFC

Problem 30 (IrMO 2010 Q2)**.** Let ABC be a triangle and let P denote the midpoint of the side BC. Suppose that there exist two points M and N interior to the sides AB and AC respectively, such that

$$|AD| = |DM| = 2|DN|$$

where D is the intersection point of the lines MN and AP. Show that $|AC| = |BC|$

Problem 31 (IrMO 2010 Q8)**.** In the triangle ABC we have $|AB| = 1$ and $\angle ABC = 120^\circ$. The perpendicular line to AB at B meets AC at D such that $|DC| = 1$. Find the length of AD.

Problem 32 (IrMO 2010 Q10)**.** Suppose a, b, c are the side lengths of a triangle ABC. Show that

$$x = \sqrt{a(b+c-a)}, \quad y = \sqrt{b(c+a-b)}, \quad z = \sqrt{c(a+b-c)}$$

are the side lengths of an acute-angled triangle XYZ, with the same area as ABC, but with a smaller perimeter, unless ABC is equilateral.

Problem 33 (IrMO 2009 Q2)**.** Let $ABCD$ be a square. The line segment AB is divided internally at H so that $|AB| \cdot |BH| = |AH|^2$. Let E be the mid point of AD and X be the midpoint of AH. Let Y be a point on EB such that XY is perpendicular to BE. Prove that $|XY| = |XH|$

Problem 34 (IrMO 2009 Q10)**.** In the triangle ABC we have $|AB| < |AC|$. The bisectors of the angles at B and C meet AC and AB at D and E respectively. BD and CE intersect at the incentre I of $\triangle ABC$

Prove that $\angle BAC = 60^\circ$ if and only if $|IE| = |ID|$

Problem 35 (IrMO 2008 Q5)**.** A triangle ABC has an obtuse angle at B. The perpendicular at B to AB meets AC at D, and $|CD| = |AB|$. Prove that $|AD|^2 = |AB| \cdot |BC|$ if and only if $\angle CBD = 30^\circ$

Problem 36 (IrMO 2008 Q7). Circles S and T intersect at P and Q, with S passing through the centre of T. Distinct points A and B lie on S, inside T, and are equidistant from the centre of T. The line PA meets T again at D. Prove that $|AD| = |PB|$.

Problem 37 (IrMO 2007 Q2). Prove that a triangle ABC is right-angled if and only if

$$\sin^2 A + \sin^2 B + \sin^2 C = 2$$

Problem 38 (IrMO 2007 Q3). The point P is a fixed point on a circle and Q is a fixed point on a line. The point R is a variable point on the circle such that P, Q and R are not collinear. The circle through P, Q and R meets the line again at V. Show that the line VR passes through a fixed point.

Problem 39 (IrMO 2007 Q8). Let ABC be a triangle the lengths of whose sides BC, CA, AB, respectively, are denoted by a, b, c, respectively. Let the internal bisectors of the angles $\angle BAC, \angle ABC, \angle BCA$ respectively, meet the sides BC, CA, AB, respectively, at D, E, F, respectively. Denote the lengths of the line segments AD, BE, CF, respectively, by d, e, f, respectively. Prove that

$$def = \frac{4abc(a+b+c)\Delta}{(a+b)(b+c)(c+a)}$$

where Δ stands for the area of the triangle ABC.

Problem 40 (IrMO 2006 Q2). P and Q are points on the equal sides AB and AC respectively of an isosceles triangle ABC such that $AP = CQ$. Moreover, neither P nor Q is a vertex of ABC. Prove that the circumcircle of the triangle APQ passes through the circumcentre of the triangle ABC.

Problem 41 (IrMO 2006 Q3). Prove that a square of side 2.1 units can be completely covered by seven squares of side 1 unit.

Problem 42 (IrMO 2006 Q7). ABC is a triangle with points D, E on BC, with D nearer B; F, G on AC, with F nearer C; H, K on AB, with H nearer A. Suppose that $AH = AG = 1$ $BK = BD = 2, CE = CF = 4, \angle B = 60^\circ$ and that D, E, F, F, G, H and K all lie on a circle. Find the radius of the incircle of the triangle ABC.

Problem 43 (IrMO 2005 Q2). Let ABC be a triangle and let D, E and F, respectively, be points on the sides BC, CA and AB, respectively-none of which coincides with a vertex of the triangle such that AD, BE and CF meet at a point G. Suppose the triangles AGF, CGE and BGD have equal area. Prove that G is the centroid of ABC.

Problem 44 (IrMO 2005 Q3). Prove that the sum of the lengths of the medians of a triangle is at least three quarters of the sum of the lengths of the sides.

Problem 45 (IrMO 2005 Q6). Let ABC be a triangle, and let X be a point on the side AB that is not A or B. Let P be the incentre of the triangle ACX, Q the incentre of the triangle BCX and M the midpoint of the segment PQ. Show that $|MC| > |MX|$.

Problem 46 (IrMO 2004 Q3). AB is a chord of length 6 of a circle centred at O and of radius 5. Let $PQRS$ denote the square inscribed in the sector OAB such that P is on the radius OA S is on the radius OB and Q and R are points on the arc of the circle between A and B. Find the area of $PQRS$.

Problem 47 (IrMO 2004 Q2)**.** A and B are distinct points on a circle $T.C$ is a point distinct from B such that $|AB| = |AC|$, and such that BC is tangent to T at B. Suppose that the bisector of $\angle ABC$ meets AC at a point D inside T. Show that $\angle ABC > 72^\circ$.

Problem 48 (IrMO 2003 Q2)**.** P, Q, R and S are (distinct) points on a circle. PS is a diameter and QR is parallel to the diameter $PS.PR$ and QS meet at A. Let O be the centre of the circle and let B be chosen so that the quadrilateral $POAB$ is a parallelogram. Prove that $BQ = BP$

Problem 49 (IrMO 2003 Q6)**.** Let T be a triangle of perimeter 2, and let a, b and c be the lengths of the sides of T

(a) Show that
$$abc + \frac{28}{27} \geq ab + bc + ac$$

(b) Show that
$$ab + bc + ac \geq abc + 1$$

Problem 50 (IrMO 2003 Q7)**.** $ABCD$ is a quadrilateral. P is at the foot of the perpendicular from D to AB, Q is at the foot of the perpendicular from D to BC, R is at the foot of the perpendicular from B to AD and S is at the foot of the perpendicular from B to CD. Suppose that $\angle PSR = \angle SPQ$. Prove that $PR = SQ$

Problem 51 (IrMO 2002 Q1)**.** In a triangle $ABC, AB = 20, AC = 21$ and $BC = 29$. The points D and E lie on the line segment BC, with $BD = 8$ and $EC = 9$. Calculate the angle $\angle DAE$

Problem 52 (IrMO 2002 Q10)**.** Let ABC be a triangle whose side lengths are all integers, and let D and E be the points at which the incircle of ABC touches BC and AC respectively. If $\left||AD|^2 - |BE|^2\right| \leq 2$, show that $|AC| = |BC|$

Problem 53 (IrMO 2001 Q2)**.** Let ABC be a triangle with sides BC, CA, AB of lengths a, b, c, respectively. Let D, E be the midpoints of the sides AC, AB, respectively. Prove that BD is perpendicular to CE if, and only if,
$$b^2 + c^2 = 5a^2$$

Problem 54 (IrMO 2001 Q8)**.** Let ABC be an acute angled triangle, and let D be the point on the line BC for which AD is perpendicular to BC. Let P be a point on the line segment AD. The lines BP and CP intersect AC and AB at E and F respectively. Prove that the line AD bisects the angle EDF

Problem 55 (IrMO 2000 Q2)**.** Let $ABCDE$ be a regular pentagon with its sides of length one. Let F be the midpoint of AB and let G, H be points on the sides CD and DE, respectively, such that $\angle GFD = \angle HFD = 30^\circ$. Prove that the triangle GFH is equilateral. A square is inscribed in the triangle GFH with one side of the square along GH. PRove that FG has length
$$t = \frac{2\cos 18^\circ \left(\cos 36^\circ\right)^2}{\cos 6^\circ}$$
and that the square has sides of length
$$\frac{t\sqrt{3}}{2 + \sqrt{3}}$$

Problem 56 (IrMO 2000 Q5)**.** Consider all parabolas of the form $y = x^2 + 2px + q(p, q$ real) which intersect the $x-$ and y -axes in three distinct points. For such a pair p, q let $C_{p,q}$ be the circle through the points of intersection of the parabola $y = x^2 + 2px + q$ with the axes. Prove that all the circles $C_{p,q}$ have a point in common.

Problem 57 (IrMO 2000 Q7)**.** Let $ABCD$ be a cyclic quadrilateral and R the radius of the circumcircle. Let a, b, c, d be the lengths of the sides of $ABCD$ and Q its area. Prove that

$$R^2 = \frac{(ab + cd)(ac + bd)(ad + bc)}{16Q^2}$$

Deduce that

$$R \geq \frac{(abcd)^{3/4}}{Q\sqrt{2}}$$

with equality if and only if $ABCD$ is a square.

Problem 58 (IrMO 1999 Q3)**.** Let D, E and F, respectively, be points on the sides BC, CA and AB, respectively, of a triangle ABC so that AD is perpendicular to BC, BE is the angle-bisector of $\angle B$ and F is the mid-point of AB. Prove that AD, BE and CF are concurrent if and only if,

$$a^2(a - c) = \left(b^2 - c^2\right)(a + c)$$

where a, b and c are the lengths of the sides BC, CA and AB, respectively, of the triangle ABC

Problem 59 (IrMO 1999 Q10)**.** $ABCDEF$ is a convex (not necessarily regular) hexagon with $AB = BC, CD = DE, EF = FA$ and

$$\angle ABC + \angle CDE + \angle EFA = 360^\circ$$

Prove that the perpendiculars from A, C and E to FB, BD and DF, respectively, are concurrent.

Problem 60 (IrMO 1998 Q2)**.** P is a point inside an equilateral triangle such that the distances from P to the three vertices are $3, 4$ and 5, respectively. Find the area of the triangle.

Problem 61 (IrMO 1998 Q4)**.** Show that a disc of radius 2 can be covered by seven (possibly overlapping) discs of radius 1

Problem 62 (IrMO 1998 Q10)**.** A triangle ABC has positive integer sides, $\angle A = 2\angle B$ and $\angle C > 90^\circ$. Find the minimum length of its perimeter.

Problem 63 (IrMO 1997 Q2)**.** Let ABC be an equilateral triangle. For a point M inside ABC, let D, E, F be the feet of the perpendiculars from M onto BC, CA, AB, respectively. Find the locus of all such points M for which $\angle FDE$ is a right-angle.

Problem 64 (IrMO 1997 Q7)**.** $ABCD$ is a quadrilateral which is circumscribed about a circle Γ (i.e., each side of the quadrilateral is tangent to Γ.) If $\angle A = \angle B = 120^\circ, \angle D = 90^\circ$ and BC has length 1, find, with proof, the length of AD.

Problem 65 (IrMO 1996 Q4)**.** Let F be the mid-point of the side BC of a triangle ABC. Isosceles right-angled triangles ABD and ACE are constructed externally on the sides AB and AC with right-angles at D and E respectively. Prove that DEF is an isosceles right-angled triangle.

Problem 66 (IrMO 1996 Q5)**.** Show, with proof, how to dissect a square into at most five pieces in such a way that the pieces can be re-assembled to form three squares no two of which are the same size.

Problem 67 (IrMO 1996 Q10)**.** Let ABC be an acute-angled triangle and let D, E, F be the feet of the perpendiculars from A, B, C onto the sides BC, CA, AB, respectively. Let P, Q, R be the feet of the perpendiculars from A, B, C onto the lines EF, FD, DE, respectively. Prove that the lines AP, BQ, CR (extended) are concurrent.

Problem 68 (IrMO 1995 Q3)**.** Let A, X, D be points on a line, with X between A and D. Let B be a point in the plane such that $\angle ABX$ is greater than $120°$, and let C be a point on the line between B and X. Prove the inequality

$$2|AD| \geq \sqrt{3}(|AB| + |BC| + |CD|)$$

Problem 69 (IrMO 1995 Q8)**.** Let S be the square consisting of all points (x, y) in the plane with $0 \leq x, y \leq 1$ For each real number t with $0 < t < 1$, let C_t denote the set of all points $(x, y) \in S$ such that (x, y) is on or above the line joining $(t, 0)$ to $(0, 1-t)$ Prove that the points common to all C_t are those points in S that are on or above the curve $\sqrt{x} + \sqrt{y} = 1$

Problem 70 (IrMO 1995 Q9)**.** We are given three points P, Q, R in the plane. It is known that there is a triangle ABC such that P is the mid-point of the side BC, Q is the point on the side CA with $|CQ|/|QA| = 2$, and R is the point on the side AB with $|AR|/|RB| = 2$ Determine, with proof, how the triangle ABC may be constructed from P, Q, R.

Problem 71 (IrMO 1994 Q2)**.** Let A, B, C be three collinear points, with B between A and C. Equilateral triangles ABD, BCE, CAF are constructed with D, E on one side of the line AC and F on the opposite side. Prove that the centroids of the triangles are the vertices of an equilateral triangle. Prove that the centroid of this triangle lies on the line AC.

Problem 72 (IrMO 1993 Q3)**.** The line l is tangent to the circle S at the point $A; B$ and C are points on l on opposite sides of A and the other tangents from B, C to S intersect at a point P. If B, C vary along l in such a way that the product $|AB| \cdot |AC|$ is constant, find the locus of P

Problem 73 (IrMO 1993 Q6)**.** Given five points P_1, P_2, P_3, P_4, P_5 in the plane having integer coordinates, prove that there is at least one pair $\left(P_i, P_j\right)$, with $i \neq j$, such that the line P_iP_j contains a point Q having integer coordinates and lying strictly between P_i and P_j.

Problem 74 (IrMO 1992 Q1)**.** Describe in geometric terms the set of points (x, y) in the plane such that x and y satisfy the condition $t^2+yt+x \geq 0$ for all t with $-1 \leq t \leq 1$

Problem 75 (IrMO 1992 Q4)**.** In a triangle ABC, the points A', B' and C' on the sides opposite A, B and C respectively, are such that the lines AA', BB' and CC' are concurrent. Prove that the diameter of the circumscribed circle of the triangle ABC equals the product $|AB'| \cdot |BC'| \cdot |CA'|$ divided by the area of the triangle $A'B'C'$

Problem 76 (IrMO 1992 Q5)**.** Let ABC be a triangle such that the coordinates of the points A and B are rational numbers. Prove that the coordinates of C are rational if, and only if, $\tan A, \tan B$ and $\tan C$, when defined, are all rational numbers.

Problem 77 (IrMO 1992 Q9)**.** A convex pentagon has the property that each of its diagonals cuts off a triangle of unit area. Find the area of the pentagon.

Problem 78 (IrMO 1991 Q1)**.** Three points X, Y and Z are given that are, respectively, the circumcentre of a triangle ABC, the mid-point of BC, and the foot of the altitude from B on AC. Show how to reconstruct the triangle ABC.

Problem 79 (IrMO 1991 Q8)**.** Let ABC be a triangle and L the line through C parallel to the side AB. Let the internal bisector of the angle at A meet the side BC at D and the line L at E, and let the internal bisector of the angle at B meet the side AC at F and the line L at G. If $|GF| = |DE|$, prove that $|AC| = |BC|$

Problem 80 (IrMO 1990 Q5)**.** Let ABC be a right-angled triangle with right-angle at A. Let X be the foot of the perpendicular from A to BC, and Y the mid-point of XC. Let AB be extended to D so that $|AB| = |BD|$. Prove that DX is perpendicular to AY

Problem 81 (IrMO 1989 Q6)**.** Suppose L is a fixed line, and A a fixed point not on L. Let k be a fixed nonzero real number. For P a point on L, let Q be a point on the line AP with $|AP|.|AQ| = k^2$ Determine the locus of Q as P varies along the line L

Problem 82 (IrMO 1988 Q1)**.** A pyramid with a square base, and all its edges of length 2, is joined to a regular tetrahedron, whose edges are also of length 2, by gluing together two of the triangular faces. Find the sum of the lengths of the edges of the resulting solid.

Problem 83 (IrMO 1988 Q2)**.** A, B, C, D are the vertices of a square, and P is a point on the arc CD of its circumcircle. Prove that

$$|PA|^2 - |PB|^2 = |PB| \cdot |PD| - |PA| \cdot |PC|$$

Problem 84 (IrMO 1988 Q3)**.** ABC is a triangle inscribed in a circle, and E is the mid-point of the arc subtended by BC on the side remote from A. If through E a diameter ED is drawn, show that the measure of the angle DEA is half the magnitude of the difference of the measures of the angles at B and C.

Problem 85 (IrMO 1988 Q4)**.** A mathematical moron is given the values b, c, A for a triangle ABC and is required to find a. He does this by using the cosine rule

$$a^2 = b^2 + c^2 - 2bc \cos A$$

and misapplying the low of the logarithm to this to get

$$\log a^2 = \log b^2 + \log c^2 - \log(2bc \cos A)$$

He proceeds to evaluate the right-hand side correctly, takes the anti-logarithms and gets the correct answer. What can be said about the triangle ABC?