IRMO INEQUALITIES

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1 Useful Inequalities

The following is a list of inequalities that are sufficient to solve every problem in the 'EGMO Problems' section of this handout. They are in a general order of usefulness.

Theorem (AM-GM Inequality). *For a collection of n non-negative real numbers a_1, a_2, ..., a_n, we have*

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \cdots a_n},$$

with equality if and only if $a_1 = a_2 = \cdots = a_n$.[1]

[1] *AM is the arithmetic mean, and GM is the geometric mean.*

Theorem (RMS-AM-GM-HM Inequality). *For n non-negative real numbers a_1, a_2, ..., a_n, the following holds.*

$$\sqrt{\frac{a_1^2 + a_2^2 + \cdots + a_n^2}{n}} \geq \frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \cdots a_n} \geq \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}}$$

with equality if and only if $a_1 = a_2 = \cdots = a_n$.[2]

[2] *RMS is the root mean squared, and HM is the harmonic mean.*

Theorem (Cauchy-Schwarz Inequality). *For the sequences of real numbers a_1, a_2, ..., a_n and b_1, b_2, ..., b_n, we have*

$$(a_1 b_1 + a_2 b_2 + \cdots + a_n b_n)^2 \leq (a_1^2 + a_2^2 + \cdots + a_n^2) \cdot (b_1^2 + b_2^2 + \cdots + b_n^2),$$

with equality if and only if $\frac{a_i}{b_i}$ is a constant for all $1 \leq i \leq n$.

Theorem (Holder's Inequality). *For sequences a_i, b_i, ..., z_i, and $\lambda_a + \lambda_b + \cdots + \lambda_z = 1$, we have*

$$a_1^{\lambda_a} b_1^{\lambda_b} \cdots z_1^{\lambda_z} + \cdots + a_n^{\lambda_a} b_n^{\lambda_b} \cdots z_n^{\lambda_z} \leq (a_1 + a_2 + \cdots + a_n)^{\lambda_a} \cdots (z_1 + z_2 + \cdots + z_n)^{\lambda_n}.$$

Theorem (Triangle Inequalty). *a, b, and c are the sidelengths of a triangle if and only if all of the following holds.*

$$b + c > a, \quad a + c > b, \text{and} \quad a + b > c$$

Theorem (Rearrangement Inequality). *For two sequences $a_1 \geq a_2 \geq \cdots \geq a_n$ and $b_1 \geq b_2 \geq \cdots \geq b_n$ then*

$$a_1 b_1 + a_2 b_2 + \cdots + a_n b_n \geq a_1 b_{\pi(1)} + a_2 b_{\pi(2)} + \cdots + a_n b_{\pi(n)} \geq a_1 b_n + a_2 b_{n-1} + \cdots + a_n b_1,$$

where $\pi(1)$, $\pi(2)$, ..., $\pi(n)$ is any permutation of $1, 2, \ldots, n$.

EGMO Problems

Problem 1 (EGMO TST 2020)**.** Let $a, b, c \geq 0$ be real numbers with $a+b+c=1$ Show that:

$$1 \leq \sqrt{a(1+b)} + \sqrt{b(1+c)} + \sqrt{c(1+a)} \leq 2$$

Hint: *For the LHS, note that* $\sqrt{a(1+b)} \geq \sqrt{a^2}$. *For the RHS, try using the condition given along Cauchy-Schwarz.*

Solution: We show the RHS first. Using the condition $a+b+c=1$,

$$\begin{aligned} &\sum_{cyc} \sqrt{a(1+b)} \leq 2 \\ \iff &\sum_{cyc} \sqrt{a}\sqrt{a+b+c+b} \leq 2 \end{aligned}$$

By Cauchy-Schwarz,

$$\begin{aligned} \sum_{cyc} \sqrt{a}\sqrt{a+b+c+b} &\leq \sqrt{a+b+c}\sqrt{(a+b+c+b)+(a+b+c+c)+(a+b+c+a)} \\ &= \sqrt{a+b+c}\sqrt{4a+4b+4c} \\ &= 2 \end{aligned}$$

as required. For the LHS, we note that $\sqrt{a(1+b)} \geq \sqrt{a^2}$, as $1+b \geq a$, thus

$$\sum_{cyc} \sqrt{a(1+b)} \geq a+b+c = 1,$$

completing the proof.

Problem 2 (EGMO TST 2019)**.** Let a, b, c be the sides of a triangle. Prove that

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} < 2$$

Hint: *Use the triangle inequality,* $b+c \geq a$ *to derive* $\frac{a}{2(b+c)} < \frac{a}{a+b+c}$, *and sum.*

Solution: As a, b, c are the sides of a triangle, $b+c > a$ and vice versa. Thus

$$\frac{a}{b+c} < \frac{2a}{a+b+c}.$$

Cycling the variables and summing yields

$$\begin{aligned} \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} &< \frac{2a}{a+b+c} + \frac{2b}{a+b+c} + \frac{2b}{a+b+c} \\ &= 2\left(\frac{a+b+c}{a+b+c}\right) \\ &= 2, \end{aligned}$$

as required.

Problem 3 (EGMO TST 2019)**.** Let $0 < x, y, z < 1$. Show that:

$$\frac{1}{x(1-y)} + \frac{1}{y(1-z)} + \frac{1}{z(1-x)} \geq 12$$

Hint: *Using AM-GM will allow you to regroup terms, resulting with* $\frac{1}{x(1-x)}$. *Another application of AM-GM then gives the required result.*

Solution: By AM-GM,

$$\sum_{cyc} \frac{1}{x(1-y)} \geq 3 \cdot \left(\frac{1}{x} \frac{1}{(1-y)} \frac{1}{y} \frac{1}{(1-z)} \frac{1}{z} \frac{1}{(1-x)} \right)^{\frac{1}{3}}$$
$$= 3 \left(\frac{1}{x(1-x)} \right)^{\frac{1}{3}} \left(\frac{1}{y(1-y)} \right)^{\frac{1}{3}} \left(\frac{1}{z(1-z)} \right)^{\frac{1}{3}}$$

Considering each term of the product and again applying AM-GM,

$$\sqrt{x(1-x)} \leq \frac{x+1-x}{2} = \frac{1}{2}$$
$$\implies x(1-x) \leq \frac{1}{4}$$
$$\implies \frac{1}{x(1-x)} \geq 4$$

Finally,

$$\sum_{cyc} \frac{1}{x(1-y)} \geq 3 \cdot 4^{\frac{1}{3}} \cdot 4^{\frac{1}{3}} \cdot 4^{\frac{1}{3}}$$
$$= 12.$$

Problem 4 (EGMO TST 2018)**.**

1. Prove that for any positive real numbers x, y we have $x^3+y^3 \geq x^2y+xy^2$
2. Prove that for any real numbers $0 \leq x, y, z \leq 1$ we have

$$3 + x^3 + y^3 + z^3 \geq x^2 + y^2 + z^2 + x + y + z$$

Hint: *For (a), either AM-GM or rearrangement will suffice. For (b), note that it is true if* $1 + x^3 \geq x^2 + x$.

Solution:

1. This is a well known inequality with a number of standard proofs.

   Using the rearrangement inequality, assume WLOG that $x \geq y$. Then,

   $$x \cdot x \cdot x + y \cdot y \cdot y \geq x \cdot x \cdot y + y \cdot y \cdot x$$

   as required.

   Alternatively, using AM-GM,

   $$2x^3 + y^3 \geq 3(x^3x^3y^3)^{\frac{1}{3}} = 3x^2y.$$

Similarly, $2y^3 + x^3 \geq 3y^2x$. Adding, we get

$$\begin{aligned} 3x^3 + 3y^3 &\geq 3x^2y + 3y^2x \\ \implies x^3 + y^3 &\geq x^2y + xy^2 \end{aligned}$$

2. Begin by noticing this is true if $1 + x^3 \geq x^2 + x$.

$$\begin{aligned} 1 + x^3 &\geq x^2 + x \\ \iff x^3 - x^2 - x + 1 &\geq 0 \\ \iff (x-1)(x+1)(x-1) &\geq 0, \end{aligned}$$

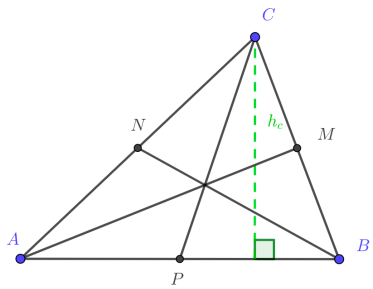
which is true, finishing the proof.

Problem 5 (EGMO TST 2017)**.** Triangle ABC has area S. Denote by M, N and P the midpoints of BC, CA and AB respectively. Prove that

$$2S\left(\frac{1}{AB} + \frac{1}{BC} + \frac{1}{CA}\right) \leq AM + BN + CP < \frac{3}{2}(AB + BC + CA)$$

Hint: *Drawing a picture will help a lot. Looking at the distances will allow you to solve both the LHS and RHS seperately.*

Solution: Consider the following diagram.



We begin by rewriting the LHS of the inequality. Let h_c denote the length of the perpendicular from a point C to the opposite side. Then,

$$\begin{aligned} 2S\left(\frac{1}{AB} + \frac{1}{BC} + \frac{1}{CA}\right) &= \left(\frac{2S}{AB} + \frac{2S}{BC} + \frac{2S}{CA}\right) \\ &= h_c + h_a + h_b. \end{aligned}$$

The shortest distance between a line and a point is the perpendicular distance, thus $h_c \leq CP$ and so on, thus

$$h_c + h_a + h_b \leq AM + BM + CP,$$

so the LHS is true.

For the RHS, we note that it is sufficient to prove that $AM < (AB + BC + CA)/2$. With this in mind, using the same principle as before (shortest distance to a line is perpendicular), we get

$$\begin{aligned} AM \le AC, \quad AM &\le AB \\ \implies 2AM &\le AC + AB \\ \implies 2AM &< AC + AB + BC \\ \implies AM &< \frac{AC + AB + BC}{2} \end{aligned}$$

This can be repeated for BN and CP, and summing the results yeilds the RHS, finishing the proof.

Problem 6 (EGMO TST 2017)**.** The positive real numbers a, b, c satisfy the double inequality

$$\frac{b^2}{a+b} + \frac{c^2}{b+c} + \frac{a^2}{c+a} \ge \frac{c^2}{a+b} + \frac{a^2}{b+c} + \frac{b^2}{c+a} \ge \frac{a^2}{a+b} + \frac{b^2}{b+c} + \frac{c^2}{c+a}$$

Prove that $a = b = c$

Hint: *It can be shown that the LHS is equal the RHS, thus comparing the LHS and the middle allows you to get terms of the form $\sum(a^2 - b^2)^2 \ge 0$, which will finish off the problem.*

Solution: This is a not-so-fun bashy problem, but we shall proceed regardless. We begin by showing the RHS and LHS are equal.

$$\begin{aligned} &\left(\frac{b^2}{a+b} + \frac{c^2}{b+c} + \frac{a^2}{c+a}\right) - \left(\frac{a^2}{a+b} + \frac{b^2}{b+c} + \frac{c^2}{c+a}\right) \\ &= \left(\frac{b^2 - a^2}{a+b}\right) + \left(\frac{c^2 - b^2}{b+c}\right) + \left(\frac{a^2 - c^2}{c+a}\right) \\ &= b^2 - a^2 + c^2 - b^2 + a^2 - c^2 \\ &= 0 \end{aligned}$$

Thus, the LHS is equal to the middle. Similar to before,

$$\begin{aligned} \left(\frac{b^2 - c^2}{a+b}\right) + \left(\frac{c^2 - a^2}{b+c}\right) + \left(\frac{a^2 - b^2}{c+a}\right) &= 0 \\ \implies \frac{(b+c)(b-c))}{a+b} + \frac{(c+a)(c-a)}{b+c} + \frac{(a-b)(a+b)}{c+a} &= 0 \\ \implies (a+b)(b+c)^2(b-c) + (a+b)(c-a)(c+a)^2 + (a+b)^2(a-b)(b+c) &= 0 \\ \implies a^2b^2 + b^2c^2 + c^2a^2 - a^4 - b^4 - c^4 &= 0 \\ \implies (a^2 - b^2)^2 + (b^2 - c^2)^2 + (c^2 - a^2)^2 &= 0 \end{aligned}$$

therefore $a = b$, $b = c$ and $a = c$.

Problem 7 (EGMO TST 2015)**.** Let x, y, z, w be positive real numbers, and suppose that $xyzw = 16$. Show that

$$\frac{x^2}{x+y} + \frac{y^2}{y+z} + \frac{z^2}{z+w} + \frac{w^2}{w+x} \ge 4$$

with equality only when $x = y = z = w = 2$

Hint: *Use Cauchy Schwarz to combine the denominators, followed by AM-GM to turn the sums into products, allowing you to use the $xyzw = 16$ condition.*

Solution: Applying Cauchy-Schwarz,

$$\begin{aligned} \sqrt{\sum_{cyc}\left(\frac{x}{\sqrt{x+y}}\right)^2} \cdot \sqrt{\sum_{cyc}(\sqrt{x+y})^2} &\geq x+y+z+w \\ \iff \sqrt{\sum_{cyc}\frac{x^2}{x+y}} \cdot \sqrt{2(x+y+z+w)} &\geq x+y+z+w \\ \iff \sum_{cyc}\frac{x^2}{x+y} &\geq \frac{x+y+z+w}{2} \end{aligned}$$

Applying AM-GM, we have

$$\begin{aligned} x+y+z+w &\geq 4(xyzw)^{\frac{1}{4}} \\ &= 4(16)^{\frac{1}{4}} \\ &= 8, \end{aligned}$$

allowing us to finish the inequality,

$$\sum_{cyc}\frac{x^2}{x+y} \geq \frac{8}{\sqrt{2}} = 4.$$

To finish the proof, we note that for equality to occur, the use of AM-GM and Cauchy-Schwarz implies that $x = y = z = w$. Then, the condition that $xyzw = 16$ forces a unique equality, when all variables are 2. Checking, we find that this is correct.

Problem 8 (EGMO TST 2014)**.** Prove that if a and b are positive real numbers,

$$\sqrt[3]{\frac{a}{b}} + \sqrt[3]{\frac{b}{a}} \leq \sqrt[3]{2(a+b)\left(\frac{1}{a}+\frac{1}{b}\right)}$$

Hint: *This is a perfect use case for Holder's inequality, a generalization of Cauchy-Schwarz!*

Solution: Using Holder's inequality, we have

$$\begin{aligned} 1^{\frac{1}{3}} \cdot a^{\frac{1}{3}} \cdot \left(\frac{1}{b}\right)^{\frac{1}{3}} + 1^{\frac{1}{3}} \cdot b^{\frac{1}{3}} \cdot \left(\frac{1}{a}\right)^{\frac{1}{3}} &\leq (1+1)^{\frac{1}{3}}(a+b)^{\frac{1}{3}}\left(\frac{1}{a}+\frac{1}{b}\right) \\ \iff \sqrt[3]{\frac{a}{b}} + \sqrt[3]{\frac{b}{a}} &\leq \sqrt[3]{2(a+b)\left(\frac{1}{a}+\frac{1}{b}\right)}. \end{aligned}$$

Problem 9 (EGMO TST 2013). Let x, y be positive integers with $3x+4y+xy = 2012$

1. Prove that $x + y \geq 83$.
2. Prove also that the same inequality is valid if 2012 is replaced by 2013.

Hint: *This looks like an inequality, but the positive integer condition makes it look a little like a number theory problem. Solving like a diophantine equation can help here.*

Solution:

1. We begin by factoring the given condition, [3]

$$\begin{aligned} 3x + 4y + xy &= 2012 \\ \implies (x+4)(y+3) - 12 &= 2012 \\ \implies (x+4)(y+3) &= 2024 \end{aligned}$$

So if $ab = 2024$ are two factors of 2024, then

$$x + 4 = a, \quad y + 3 = b \quad \implies \quad x + y = a + b - 7$$

thus it is sufficient to show there is no factor pairs with a sum less than 90. $2024 = 2^3 \times 11 \times 23$, and using rearrangement we find that $23 \cdot 2 + 11 \cdot 2 \cdot 2 = 90$ is minium, so this is true.

2. Replacing 2012 with 2013, we again find that it's sufficient to show there is no factor pairs with a sum of less than 90. Factoring, $2013 = 3 \times 11 \times 61$, thus the minimum sum is $61 + 11 \cdot 3 = 94$, so this is also true.

[3] *This factoring method is sometimes called 'Simons Favourite Factoring Trick', from the early years of the AoPS website.*

Problem 10 (EGMO TST 2012). Prove that for all positive real numbers x and y satisfying $x + y = 1$ the following inequality holds

$$\left(x + \frac{1}{x}\right)^2 + \left(y + \frac{1}{y}\right)^2 \geq \frac{25}{2}.$$

Hint: *The squares are vaguelly reminiscent of RMS, and the $1/x$ and $1/y$ terms are reminiscent of HM, so such inequalities may be helpful.*

Solution: With the intention of applying AM-RMS, we rewrite the inequality as the equivalent

$$\sqrt{\frac{\left(x + \frac{1}{x}\right)^2 + \left(y + \frac{1}{y}\right)^2}{2}} \geq \frac{5}{2}.$$

Then, by AM-RMS,

$$\sqrt{\frac{\left(x + \frac{1}{x}\right)^2 + \left(y + \frac{1}{y}\right)^2}{2}} \geq \frac{x + y + \frac{1}{x} + \frac{1}{y}}{2} = \frac{1 + \frac{1}{x} + \frac{1}{y}}{2},$$

thus we must prove $\frac{1}{x}+\frac{1}{y}\geq 4$. This is straightforward using AM-HM,

$$\begin{aligned} \frac{2}{\frac{1}{x}+\frac{1}{y}} &\leq \frac{x+y}{2}=\frac{1}{2} \\ \iff \frac{1}{\frac{1}{x}+\frac{1}{y}} &\leq \frac{1}{4} \\ \iff \frac{1}{x}+\frac{1}{y} &\geq 4, \end{aligned}$$

finishing the proof.

Past Problems

This is a collection of all inequality problems that have appeared in the Irish Mathematical Olympiad. The questions are ordered chronologically. All problems are due to their respective creators.

Problem 1 (IrMO 2019)**.** Supose x, y, z are real mumbers such that $x^2+y^2+z^2+2xyz=1$. Prove that $8xyz \leq 1$ with equality if and only if (x, y, z) is one of the following:

$$\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right), \quad \left(-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right), \quad \left(-\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right), \quad \left(\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right)$$

Problem 2 (IrMO 2018)**.** Let a, b, c be the side lengths of a triangle. Prove that

$$2\left(a^3+b^3+c^3\right)<(a+b+c)\left(a^2+b^2+c^2\right) \leq 3\left(a^3+b^3+c^3\right)$$

Problem 3 (IrMO 2017)**.** Show that for all non-negative numbers a, b

$$1+a^{2017}+b^{2017} \geq a^{10} b^7+a^7 b^{2000}+a^{2000} b^{10}$$

When is equality attained?

Problem 4 (IrMO 2016)**.** Let $a_1, a_2, \ldots, a_m$ be positive integers, none of which is equal to 10, such that $a_1+a_2+\cdots+a_m=10m$. Prove that

$$\left(a_1 a_2 a_3 \cdots a_m\right)^{1/m} \leq 3 \sqrt{11}$$

Problem 5 (IrMO 2015)**.** Suppose x, y are nonnegative real numbers such that $x+y \leq 1$. Prove that

$$8xy \leq 5x(1-x)+5y(1-y)$$

eand determine the cases of equality.

Problem 6 (IrMO 2015)**.** Prove that, for all pairs of nonnegative integers, j, n

$$\sum_{k=0}^{n} k^j\binom{n}{k} \geq 2^{n-j} n^j$$

Problem 7 (IrMO 2014)**.** Suppose $a_1, \dots, a_n > 0$, where $n > 1$ and $\sum_{i=1}^{n} a_i = 1$. For each $i = 1, 2, \dots, n$, let $b_i = a_i^2 / \sum_{j=1}^{n} a_j^2$ Prove that

$$\sum_{i=1}^{n} \frac{a_i}{1 - a_i} \leq \sum_{i=1}^{n} \frac{b_i}{1 - b_i}$$

When does equality occur?

Problem 8 (IrMO 2013)**.** Prove that

$$1 - \frac{1}{2012}\left(\frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{2013}\right) \geq \frac{1}{\sqrt[2012]{2013}}$$

Problem 9 (IrMO 2013)**.** Let a, b, c be real numbers and let

$$x = a + b + c, y = a^2 + b^2 + c^2, z = a^3 + b^3 + c^3 \quad \text{and} \quad S = 2x^3 - 9xy + 9z$$

(a) Prove that S is unchanged when a, b, c are replaced by $a + t, b + t, c + t$, respectively, for any real number t (b) Prove that $\left(3y - x^2\right)^3 \geq 2S^2$

Problem 10 (IrMO 2012)**.** Suppose a, b, c are positive numbers. Prove that

$$\left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a} + 1\right)^2 \geq (2a + b + c)\left(\frac{2}{a} + \frac{1}{b} + \frac{1}{c}\right)$$

with equality if and only if $a = b = c$

Problem 11 (IrMO 2012)**.** (a) Show that if x and y are positive real numbers, then

$$(x + y)^5 \geq 12xy\left(x^3 + y^3\right)$$

(b) Prove that the constant 12 is the best possible. In other words, prove that for any $K > 12$ there exist positive real numbers x and y such that

$$(x + y)^5 < Kxy\left(x^3 + y^3\right)$$

Problem 12 (IrMO 2011)**.** Suppose that x, y and z are positive numbers such that

$$1 = 2xyz + xy + yz + zx$$

Prove that

(a)
$$\frac{3}{4} \leq xy + yz + zx < 1$$

(b)
$$xyz \leq \frac{1}{8}$$

thus, deduce that
$$x + y + z \geq \frac{3}{2} \tag{1}$$

and derive the case of equality in Equation 1.

Problem 13 (IrMO 2010)**.** Suppose x, y, z are positive numbers such that $x + y + z = 1$. Prove that

(a) $xy + yz + zx \geq 9xyz$

(b) $xy + yz + zx < \frac{1}{4} + 3xyz$

Problem 14 (IrMO 2009)**.** Suppose a, b, c are real numbers such that $a+b+c=0$ and $a^2+b^2+c^2=1$. Prove that
$$a^2b^2c^2 \leq \frac{1}{54}$$
and determine the cases of equality.

Problem 15 (IrMO 2009)**.** Suppose that x, y and z are positive real numbers such that $xyz \geq 1$

- Prove that
$$27 \leq (1+x+y)^2 + (1+y+z)^2 + (1+z+x)^2$$
with equality if and only if $x = y = z = 1$
- Prove that
$$(1+x+y)^2 + (1+y+z)^2 + (1+z+x)^2 \leq 3(x+y+z)^2$$
with equality if and only if $x = y = z = 1$

Problem 16 (IrMO 2008)**.** For positive real numbers a, b, c and d such that $a^2+b^2+c^2+d^2=1$ prove that
$$a^2b^2cd + ab^2c^2d + abc^2d^2 + a^2bcd^2 + a^2bc^2d + ab^2cd^2 \leq \frac{3}{32}$$
and determine the cases of equality.

Problem 17 (IrMO 2007)**.** Suppose a, b and c are positive real numbers. Prove that
$$\frac{a+b+c}{3} \leq \sqrt{\frac{a^2+b^2+c^2}{3}} \leq \frac{\frac{ab}{c} + \frac{bc}{a} + \frac{ca}{b}}{3}$$
For each of the inequalities, find conditions on a, b and c such that equality holds.

Problem 18 (IrMO 2006)**.** Suppose x and y are positive real numbers such that $x+2y=1$. Prove that
$$\frac{1}{x} + \frac{2}{y} \geq \frac{25}{1+48xy^2}$$

Problem 19 (IrMO 2005)**.** Suppose a, b and c are non-negative real numbers. Prove that $\frac{1}{3}\left[(a-b)^2+(b-c)^2+(c-a)^2\right] \leq a^2+b^2+c^2-3\sqrt[3]{a^2b^2c^2} \leq (a-b)^2+(b-c)^2+(c-a)^2$

Problem 20 (IrMO 2004)**.** Define the function m of the three real variables x, y, z by
$$m(x,y,z) = \max\left(x^2, y^2, z^2\right), x, y, z \in \mathbb{R}$$
Determine, with proof, the minimum value of m if x, y, z vary in $\mathbb{R}$ subject to the following restrictions:
$$x+y+z=0, \quad x^2+y^2+z^2=1$$

Problem 21 (IrMO 2004)**.** Let $a, b \geq 0$. Prove that
$$\sqrt{2}(\sqrt{a(a+b)^3} + b\sqrt{a^2+b^2}) \leq 3\left(a^2+b^2\right)$$
with equality if and only if $a = b$

Problem 22 (IrMO 2003)**.** Let T be a triangle of perimeter 2, and let a, b and c be the lengths of the sides of T.

(a) Show that
$$abc + \frac{28}{27} \geq ab + bc + ac$$

(b) Show that
$$ab + bc + ac \geq abc + 1$$

Problem 23 (IrMO 2002)**.** Let $0 < a, b, c < 1$. Prove that
$$\frac{a}{1-a} + \frac{b}{1-b} + \frac{c}{1-c} \geq \frac{3\sqrt[3]{abc}}{1-\sqrt[3]{abc}}$$
Determine the case of equality.

Problem 24 (IrMO 2001)**.** Prove that

(a) $\frac{2n}{3n+1} \leq \sum_{k=n+1}^{2n} \frac{1}{k}$

(b) $\sum_{k=n+1}^{2n} \frac{1}{k} \leq \frac{3n+1}{4(n+1)}$

for all positive integers n

Problem 25 (IrMO 2000)**.** Let $ABCD$ be a cyclic quadrilateral and R the radius of the circumcircle. Let a, b, c, d be the lengths of the sides of $ABCD$ and Q its area. Prove that
$$R^2 = \frac{(ab+cd)(ac+bd)(ad+bc)}{16Q^2}$$
Deduce that
$$R \geq \frac{(abcd)^{3/4}}{Q\sqrt{2}}$$
with equality if and only if $ABCD$ is a square.

Problem 26 (IrMO 2000)**.** Let $x \geq 0, y \geq 0$ be real numbers with $x + y = 2$. Prove that
$$x^2y^2\left(x^2 + y^2\right) \leq 2$$

Problem 27 (IrMO 1999)**.** Let a, b, c and d be positive real numbers whose sum is 1. Prove that
$$\frac{a^2}{a+b} + \frac{b^2}{b+c} + \frac{c^2}{c+d} + \frac{d^2}{d+a} \geq \frac{1}{2}$$
with equality if, and only if, $a = b = c = d = 1/4$

Problem 28 (IrMO 1999)**.** Find all real values x that satisfy
$$\frac{x^2}{(x+1-\sqrt{x+1})^2} < \frac{x^2+3x+18}{(x+1)^2}$$

Problem 29. Prove that if a, b, c are positive real numbers, then

(a) $\frac{9}{a+b+c} \leq 2\left(\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a}\right)$

(b) $\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} \leq \frac{1}{2}\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)$

Problem 30 (IrMO 1998)**.** Show that if x is a nonzero real number, then
$$x^8 - x^5 - \frac{1}{x} + \frac{1}{x^4} \geq 0$$

Problem 31 (IrMO 1997)**.** Suppose a, b and c are nonnegative real numbers such that $a+b+c \geq abc$. Prove that $a^2+b^2+c^2 \geq abc$

Problem 32 (IrMO 1996)**.** Prove that the inequality

$$2^{\frac{1}{2}} \cdot 4^{\frac{1}{4}} \cdot 8^{\frac{1}{8}} \cdots (2^n)^{\frac{1}{2^n}} < 4$$

holds for all positive integers n

Problem 33 (IrMO 1995)**.** Prove the inequalities $n^n \leq (n!)^2 \leq [(n+1)(n+2)/6]^n$ for every positive integer n

Problem 34 (IrMO 1994)**.** Prove that, for every integer $n > 1$

$$n\left((n+1)^{2/n} - 1\right) < \sum_{i=1}^{n} \frac{2i+1}{i^2} < n\left(1 - n^{-2/(n-1)}\right) + 4$$

Problem 35 (IrMO 1994)**.** Let $f(n)$ be defined on the set of positive integers by the rules: $f(1) = 2$ and

$$f(n+1) = (f(n))^2 - f(n) + 1, \quad n = 1, 2, 3, \ldots$$

Prove that, for all integers $n > 1$

$$1 - \frac{1}{2^{2n-1}} < \frac{1}{f(1)} + \frac{1}{f(2)} + \ldots + \frac{1}{f(n)} < 1 - \frac{1}{2^{2^n}}$$

Problem 36 (IrMO 1992)**.** Let a, b, c and d be real numbers with $a \neq 0$. Prove that if all the roots of the cubic equation

$$az^3 + bz^2 + cz + d = 0$$

lie to the left of the imaginary axis in the complex plane, then

$$ab > 0, bc - ad > 0, ad > 0$$

Problem 37 (IrMO 1992)**.** If, for $k = 1, 2, \ldots, n$, a_k and b_k are positive real numbers, prove that

$$\sqrt[n]{a_1 a_2 \cdots a_n} + \sqrt[n]{b_1 b_2 \cdots b_n} \leq \sqrt[n]{(a_1+b_1)(a_2+b_2)\cdots(a_n+b_n)}$$

and that equality holds if, and only if,

$$\frac{a_1}{b_1} = \frac{a_2}{b_2} = \cdots = \frac{a_n}{b_n}$$

Problem 38 (IrMO 1992)**.** Describe in geometric terms the set of points (x, y) in the plane such that x and y satisfy the condition $t^2 + yt + x \geq 0$ for all t with $-1 \leq t \leq 1$

Problem 39 (IrMO 1991)**.** Let

$$a_n = \frac{n^2+1}{\sqrt{n^4+4}}, \quad n = 1, 2, 3, \ldots$$

and let b_n be the product of $a_1, a_2, a_3, \dots, a_n$. Prove that

$$\frac{b_n}{\sqrt{2}} = \frac{\sqrt{n^2+1}}{\sqrt{n^2+2n+2}}$$
$$\text{and deduce that} \quad \frac{1}{n^3+1} < \frac{b_n}{\sqrt{2}} - \frac{n}{n+1} < \frac{1}{n^3}$$

for all positive integers n

Problem 40 (IrMO 1990)**.** The real number x satisfies all the inequalities

$$2^k < x^k + x^{k+1} < 2^{k+1}$$

for $k = 1, 2, \dots, n$. What is the greatest possible value of n?

Problem 41 (IrMO 1989)**.** Suppose P is a point in the interior of a triangle ABC, that x, y, z are the distances from P to A, B, C, respectively, and that p, q, r are the perpendicular distances from P to the sides BC, CA, AB respectively. Prove that $xyz \geq 8pqr$ with equality implying that the triangle ABC is equilateral.

Problem 42 (IrMO 1988)**.** Let $0 \leq x \leq 1$. Show that if n is any positive integer, then

$$(1+x)^n \geq (1-x)^n + 2nx\left(1-x^2\right)^{\frac{n-1}{2}}$$